Symbolic Computation for Integro-Differential-Time-Delay Operators with Matrix Coefficients ⋆

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Abstract: The purpose of this paper is to algebraize and automatize computations with linear differential time-delay systems and their solutions. To this end, we explain an algebraic construction of the ring of integro-differential operators with linear substitutions having (noncommutative) matrix coefficients, which contains the ring of integro-differential-time-delay operators. Based on a reduction system for this ring, we show how such operators can be uniquely expanded into irreducible terms.
Symbolic computations with these operators and their normal forms are implemented in a `Mathematica` package. This even allows for computations with systems having generic size and/or undetermined matrix coefficients. We illustrate how, by elementary computations with operators in this framework, results like the method of steps can be found and proven in an automated way. Normal form computations with our package can be used to partly automatize solving operator equations. As an example, we recover a generalization of Artstein's reduction, which solves an equivalence problem of a class of differential time-delay control systems.

Keywords: Differential time-delay systems, computer algebra, integro-differential operators with linear substitutions, normal forms, Artstein's reduction, algebraic analysis approach to linear systems theory

1. INTRODUCTION

The goal of the paper is to algebraize and automatize symbolic manipulations of linear differential time-delay (DTD) systems like

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h), \tag{1}$$

where $A_0(t)$ and $A_1(t)$ are square matrices and $h > 0$. When the matrices $A_0(t)$ and $A_1(t)$ in (1) are explicitly given, a standard approach utilizes the ring of DTD operators $\mathcal{D} = \mathcal{A}\langle\partial, \delta\rangle$ with a coefficient ring $\mathcal{A}$ of scalar functions. This ring consists of all sums of terms of the form $f\delta^i\partial^j$, where ∂ and δ stand for the operators mapping $f(t)$ to $\dot{f}(t)$ and $f(t-h)$, respectively. The system (1) can be represented by the matrix of DTD operators

$$R = \partial I_n - A_0 - A_1\delta \in \mathcal{D}^{n\times n},$$

where, for shorter notation, coefficients are collected into matrices $A_0, A_1 \in \mathcal{A}^{n\times n}$. Within this so-called *algebraic analysis* approach, the system can be studied by means of R and of the properties of the corresponding left $\mathcal{D}$-module, see Section 2 for details.

When studying certain (control) problems, it sometimes is necessary to consider not only a fixed linear differential time-delay system but whole classes of systems as, for instance, the set of all systems of the form (1) where the matrices A_0 and A_1 are general matrices of generic size. We are then led to perform formal computations with undetermined matrices as coefficients. To this end, we directly equip operators with coefficients from some ring of matrices $\mathcal{R}$, i.e., we consider the ring of operators $\mathcal{R}\langle\partial, \delta\rangle$. Then, the above system is represented by the operator

$$\partial - A_0 - A_1 \cdot \delta \in \mathcal{R}\langle\partial, \delta\rangle$$

with actual matrices $A_0, A_1 \in \mathcal{R}$ as coefficients. In this way, general classes of systems can be considered at once and results on these classes can be obtained directly. To this end, computer algebra methods for rings of operators with matrix coefficients are needed.

In Section 3, we explain the ring $\mathcal{R}\langle\partial, \textstyle\int, \mathrm{E}, \mathcal{S}\rangle$ of integro-differential operators with linear substitutions (IDOLS) formalized in (Hossein Poor et al., 2018). An important motivation for studying this ring, which contains $\mathcal{R}\langle\partial, \delta\rangle$, comes from the work by Quadrat (2015). In that paper, such operators and their commutation rules were already

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used for an algebraic approach to obtain Artstein's integral transformation of linear DTD control systems (Artstein, 1982). Based on our package `TenReS` (Hossein Poor et al., 2016), we implemented IDOLS in the computer algebra system `Mathematica`. This implementation, along with all computations in this paper, is available at:

`http://gregensburger.com/softw/idols`

Using the algebraic framework for IDOLS and its implementation, classes of linear DTD systems and their transformations can be studied. Operator equations can be solved by ansatz and normal form computations. Moreover, by formal computations in the ring $\mathcal{R}\langle\partial, \int, \mathrm{E}, \mathcal{S}\rangle$, also operators having rectangular matrices as their coefficients can be treated. To illustrate these use-cases, in Section 4, we use the implementation of IDOLS to largely automatize the computations of (Quadrat, 2015) to recover Artstein's transformation. The implementation allows to study more involved problems and find new results that cannot easily be obtained by hand computations. In a forthcoming publication, we will present other applications of this computer algebra approach to the study of classes of linear DTD systems and related equivalence problems.

Throughout this paper, rings are not necessarily commutative, but they are always assumed to have a unit element (of multiplication). All our operators act from the left on some module of "functions". Furthermore, we use operator notation, e.g., we write δA instead of $\delta(A)$ and $\partial AB = (\partial A)B + A\partial B$ for $\partial(AB) = \partial(A)B + A\partial(B)$.

2. HOMOMORPHISMS OF LINEAR FUNCTIONAL SYSTEMS

In this section, we recall the characterization of the transformations which map solutions of a linear functional system (e.g., differential, time-delay, difference, ...) to solutions of another one. To do so, we use the so-called *algebraic analysis approach* which provides a unified mathematical framework for studying linear systems of functional equations. For more details, see (Chyzak et al., 2005) and the references therein. Within this approach, a rectangular system of q linear functional equations in p unknown functions is defined by means of a $q \times p$ matrix with entries in a noncommutative ring $\mathcal{D}$ of functional operators. If $\mathcal{F}$ is a left $\mathcal{D}$-module, e.g., a functional space which is closed under the left action of $\mathcal{D}$, then a linear system, also called *behavior*, can be defined as follows:

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

A transformation between systems defined by $R \in \mathcal{D}^{q\times p}$ and $R' \in \mathcal{D}^{q'\times p'}$ maps a solution $\eta' \in \ker_{\mathcal{F}}(R'.)$ to a solution $\eta \in \ker_{\mathcal{F}}(R.)$. If we can find a matrix $P \in \mathcal{D}^{p\times p'}$ for which there exists a matrix $Q \in \mathcal{D}^{q\times q'}$ satisfying

$$R\,P = Q\,R', \tag{2}$$

then P defines such a transformation by $\eta = P\eta'$ since $R\,\eta = R\,(P\,\eta') = Q\,(R'\,\eta') = 0$ for all $\eta' \in \ker_{\mathcal{F}}(R'.)$.

In the following, we explain the algebraic background of the algebraic analysis approach for the interested reader. A behavior is an analytic object which can be studied by means of algebraic techniques (e.g., module theory, homological algebra) by considering the left $\mathcal{D}$-module

$$M := \mathcal{D}^{1\times p}/(\mathcal{D}^{1\times q}\,R)$$

finitely presented by the matrix $R \in \mathcal{D}^{q\times p}$. Indeed, in homological algebra, a standard result asserts that we have an isomorphism of abelian groups

$$\ker_{\mathcal{F}}(R.) \cong \hom_{\mathcal{D}}(M, \mathcal{F}),$$

where $\hom_{\mathcal{D}}(M, \mathcal{F})$ is the abelian group of all left $\mathcal{D}$-homomorphisms $\phi : M \longrightarrow \mathcal{F}$, i.e. $\phi(d_1\,m_1 + d_2\,m_2) = d_1\,\phi(m_1) + d_2\,\phi(m_2)$ for all $d_1, d_2 \in \mathcal{D}$ and $m_1, m_2 \in M$.

A dictionary between the built-in properties of the behavior $\ker_{\mathcal{F}}(R.)$ and the algebraic properties of the finitely presented left $\mathcal{D}$-module M can then be developed. Moreover, the algebraic properties of M, and thus those of the behavior $\ker_{\mathcal{F}}(R.)$, can be effectively checked using computer algebra techniques (e.g., Gröbner/Janet basis computations over noncommutative polynomial rings). For more details, see (Chyzak et al., 2005; Cluzeau and Quadrat, 2008; Quadrat, 2015) and references therein.

For two behaviors $\ker_{\mathcal{F}}(R.)$ and $\ker_{\mathcal{F}}(R'.)$ as above, we consider the left $\mathcal{D}$-modules M and M' finitely presented by $R \in \mathcal{D}^{q\times p}$ and $R' \in \mathcal{D}^{q'\times p'}$, respectively. The following theorem shows that every left $\mathcal{D}$-homomorphism $\phi : M \longrightarrow M'$ induces an abelian group homomorphism $\phi^\star : \ker_{\mathcal{F}}(R'.) \longrightarrow \ker_{\mathcal{F}}(R.)$, i.e., a transformation which sends a solution of $R'\,\eta' = 0$ to a solution of $R\,\eta = 0$.

Theorem 1. (Cluzeau and Quadrat (2008)). With the notations explained above, we have:

(1) A left $\mathcal{D}$-homomorphism $\phi : M \longrightarrow M'$ is defined by
$$\phi(\pi(\lambda)) = \pi'(\lambda\,P), \quad \forall \lambda \in \mathcal{D}^{1\times p},$$
where $\pi : \mathcal{D}^{1\times p} \longrightarrow M$ (resp., $\pi' : \mathcal{D}^{1\times p'} \longrightarrow M'$) denotes the canonical projection onto M (resp., M'), and the matrix $P \in \mathcal{D}^{p\times p'}$ is such that there exists a matrix $Q \in \mathcal{D}^{q\times q'}$ satisfying $R\,P = Q\,R'$.

(2) A left $\mathcal{D}$-homomorphism $\phi : M \longrightarrow M'$ induces the following homomorphism of abelian groups:
$$\begin{aligned}\phi^\star : \ker_{\mathcal{F}}(R'.) &\longrightarrow \ker_{\mathcal{F}}(R.)\\ \eta' &\longmapsto \eta := P\,\eta'.\end{aligned}$$

With the notations of Theorem 1, if the matrix P is a square invertible matrix, then the transformation $\phi^\star$ defined by P is invertible and its inverse $\phi^{\star -1}$, which maps solutions of $R\,\eta = 0$ to solutions of $R'\,\eta' = 0$, is then defined by P^{-1}. The effective computation of $\phi^\star$ relies on the resolution of (2). The feasibility/difficulty of this task heavily depends on the ring $\mathcal{D}$. When $\mathcal{D}$ belongs to a certain class of Ore algebras of functional operators, we refer to (Cluzeau and Quadrat, 2008) for algorithms solving (2). In Section 4, we show by an example that the formalism presented in Section 3 allows us to solve (2) in the ring of IDOLS by ansatz.

3. INTEGRO-DIFFERENTIAL-DELAY OPERATORS

Rings of integro-differential operators over a field of constants were introduced in (Rosenkranz, 2005; Rosenkranz and Regensburger, 2008) to study algebraic and algorithmic aspects of linear ordinary boundary problems. See (Regensburger, 2016) for an overview and related references. In (Quadrat, 2015), integro-differential-time-delay operators and their algebraic commutation rules were introduced. In (Hossein Poor et al., 2018), integro-differential operators with linear substitutions (IDOLS) over noncommutative coefficient rings were formally defined and their normal forms were worked out based on tensor reduction systems. In Sections 3.2 and 3.3 below, we explain the definition of and computations in the ring of IDOLS. This ring provides an algebraic setting for dealing with DTD systems and corresponding initial value problems in general, as illustrated by the examples below.

3.1 Integro-Differential Rings with Linear Substitutions

In the following, we introduce an algebraic structure to (matrices of) coefficients and the operations differentiation, integration, evaluation, and linear substitution acting on them. For an easy understanding, one can think of the matrix ring $\mathcal{R} = C^\infty(\mathbb{R})^{n\times n}$ as a concrete instance. For matrices, the Leibniz rule takes the form

$$\tfrac{d}{dt}(A(t)B(t)) = \dot{A}(t)B(t) + A(t)\dot{B}(t)$$

and the constants are given by the noncommutative ring $\mathbb{R}^{n\times n}$. In general, we have the following definition.

Definition 2. Let $\mathcal{R}$ be a ring and let $\partial : \mathcal{R} \longrightarrow \mathcal{R}$ be an additive map satisfying the *Leibniz rule*

$$\partial AB = (\partial A)B + A\partial B. \tag{3}$$

Then, $(\mathcal{R}, \partial)$ is called a *differential ring* and its *ring of constants* is given by

$$\{C \in \mathcal{R} \mid \partial C = 0\}.$$

Note that $\mathcal{R}$ and the ring of constants $\mathcal{K}$ of $(\mathcal{R}, \partial)$ are not necessarily commutative rings. The ring $\mathcal{R}$ is a bimodule over $\mathcal{K}$ since it has both a left and a right $\mathcal{K}$-module structure which satisfy $(C_1A)C_2 = C_1(AC_2)$ for all $C_1, C_2 \in \mathcal{K}$ and for all $A \in \mathcal{R}$. From the properties of ∂, it follows that ∂ is both left and right linear over the ring of constants $\mathcal{K}$. In short, $\mathcal{R}$ is a $\mathcal{K}$-bimodule and ∂ is a $\mathcal{K}$-bimodule endomorphism.

Likewise, integration $\int_{t_0}^t A(s)\,ds$ of matrices $A(t)$ induces a bimodule endomorphism over the ring of constant matrices. By the fundamental theorem of calculus, we have

$$\tfrac{d}{dt}\int_{t_0}^t A(s)\,ds = A(t)$$

and the evaluation at t_0 can be expressed in terms of differentiation and integration as follows:

$$A(t_0) = A(t) - \int_{t_0}^t \dot{A}(s)\,ds.$$

Moreover, the evaluation at t_0 of a product is the product of the individual evaluations. In other words, evaluation at t_0 is a multiplicative operation. These properties motivate the following general definition.

Definition 3. Let $(\mathcal{R}, \partial)$ be a differential ring such that $\partial\mathcal{R} = \mathcal{R}$ and let $\mathcal{K}$ be its ring of constants. Moreover, let $\int : \mathcal{R} \longrightarrow \mathcal{R}$ be a $\mathcal{K}$-bimodule endomorphism such that for all $A \in \mathcal{R}$ we have

$$\partial{\textstyle\int} A = A. \tag{4}$$

Then, $(\mathcal{R}, \partial, \int)$ is called an *integro-differential ring* if the *evaluation* $\mathrm{E} : \mathcal{R} \longrightarrow \mathcal{R}$ defined by

$$\mathrm{E}A := A - {\textstyle\int}\partial A \tag{5}$$

is multiplicative, i.e., for all $A, B \in \mathcal{R}$, we have

$$\mathrm{E}AB = (\mathrm{E}A)\mathrm{E}B.$$

Using (4) and (5), it can be checked easily that the evaluation E surjectively maps $\mathcal{R}$ onto $\mathcal{K}$.

In order to incorporate time-delays and dilations, we consider the more general case of linear substitutions. More formally, let $\sigma_{a,b}$ denote the linear substitution operator mapping $A(t)$ to $A(at - b)$ for a nonzero $a \in \mathbb{R}$ and an arbitrary $b \in \mathbb{R}$. This operation is multiplicative and by the chain rule we have

$$\tfrac{d}{dt}(\sigma_{a,b}A(t)) = a\dot{A}(at - b) = a\sigma_{a,b}\dot{A}(t).$$

In the case of matrices of smooth functions, $\mathbb{R}$ can be identified with $\{rI_n \mid r \in \mathbb{R}\}$, which is precisely the set of constant matrices that commute with all matrices. The time-delay operator mapping $A(t)$ to $A(t - h)$ is given by

$$\delta = \sigma_{1,h},$$

which is used throughout the paper.

Definition 4. Let $(\mathcal{R}, \partial, \int)$ be an integro-differential ring and let $\mathcal{K}$ be its ring of constants. Let $\mathcal{C}$ be the ring of elements of $\mathcal{K}$ that commute with all elements of $\mathcal{R}$ and let $\mathcal{C}^*$ denote its group of units. Moreover, let

$$\mathcal{S} := \{\sigma_{a,b} \mid a \in \mathcal{C}^*, b \in \mathcal{C}\},$$

where $\sigma_{a,b} : \mathcal{R} \longrightarrow \mathcal{R}$ are multiplicative $\mathcal{K}$-bimodule endomorphisms of $\mathcal{R}$ fixing the constants $\mathcal{K}$, be such that for all $a, c \in \mathcal{C}^*$, $b, d \in \mathcal{C}$, and $A \in \mathcal{R}$ we have

$$\begin{gathered}\sigma_{1,0}A = A, \quad \sigma_{a,b}\sigma_{c,d}A = \sigma_{ac,bc+d}A, \quad \text{and}\\ \partial\sigma_{a,b}A = a\sigma_{a,b}\partial A.\end{gathered} \tag{6}$$

Then, $(\mathcal{R}, \partial, \int, \mathcal{S})$ is called an *integro-differential ring with linear substitutions.*

In practice, it usually suffices to do computations in the free integro-differential ring with linear substitutions generated by the expressions occurring in the systems under consideration. Whenever more specific relations among the generators are known, computations are done modulo additional identities taken from the integro-differential ideal generated by those relations. This is the approach taken in all our examples and in our package. Formally, the free ring above is constructed by considering the term algebra on the set of generators modulo the identities that hold in any integro-differential ring with linear substitutions. See, for example, (Baader and Nipkow, 1998, Ch. 3) or (Cohn, 2003, Ch. 1) for details on the general construction of free algebraic structures in universal algebra.

Example 5. To model a fundamental system of the equation $\dot{x}(t) - A_0(t)x(t) = 0$, we assume $\Phi \in \mathcal{R}$ is invertible and satisfies $\partial\Phi - A_0\Phi = 0$. In order to compute with this Φ, we consider the free integro-differential ring with linear substitutions generated by the symbols A_0, Φ, Φ^{-1}. We consider a factor ring of this free ring by computing modulo the additional identities $\partial\Phi - A_0\Phi = 0$, $\Phi\Phi^{-1} = 1$, $\Phi^{-1}\Phi = 1$, and $\partial(\Phi^{-1}) = -\Phi^{-1}(\partial\Phi)\Phi^{-1}$. Thereby, we can compute $\partial(\Phi^{-1}) = -\Phi^{-1}A_0$. Hence, this identity also holds in $\mathcal{R}$.

3.2 Integro-Differential Operators with Linear Substitutions

In the following, starting from a given integro-differential ring with linear substitutions $(\mathcal{R}, \partial, \int, \mathcal{S})$, we construct the corresponding ring of operators generated by $\partial, \int, \mathrm{E}$, and the elements of $\mathcal{R}$ and $\mathcal{S}$. This ring has a natural action on $\mathcal{R}$, where the elements of $\mathcal{R}$ act as multiplication operators and $\partial, \int, \mathrm{E}$, and the elements of $\mathcal{S}$ act as the corresponding operations.

Definition 6. Let $(\mathcal{R}, \partial, \int, \mathcal{S})$ be an integro-differential ring with linear substitutions and let $\mathcal{K}$ be its ring of constants. We let

$$\mathcal{R}\langle\partial, \textstyle\int, \mathrm{E}, \mathcal{S}\rangle$$

be the ring generated by $\mathcal{R}$ and $\partial, \int, \mathrm{E}, \mathcal{S}$, where

(1) the identities of Table 1 hold for all $A, B \in \mathcal{R}$ and $\sigma_{a,b}, \sigma_{c,d} \in \mathcal{S}$ and
(2) $\partial, \int, \mathrm{E}$, and each element of $\mathcal{S}$ commute with all elements of $\mathcal{K}$.

We call $\mathcal{R}\langle\partial, \int, \mathrm{E}, \mathcal{S}\rangle$ the *ring of integro-differential operators with linear substitutions (IDOLS).*

Table 1. Operator Identities

$A \cdot B = AB$	$\sigma_{1,0} = 1$
$\partial \cdot A = A \cdot \partial + \partial A$	$\sigma_{a,b} \cdot \sigma_{c,d} = \sigma_{ac,bc+d}$
$\mathrm{E} \cdot A = (\mathrm{E}A)\mathrm{E}$	$\sigma_{a,b} \cdot A = \sigma_{a,b}A \cdot \sigma_{a,b}$
$\partial \cdot \int = 1$	$\sigma_{a,b} \cdot \mathrm{E} = \mathrm{E}$
$\int \cdot \partial = 1 - \mathrm{E}$	$\partial \cdot \sigma_{a,b} = a\sigma_{a,b} \cdot \partial$

The multiplication in $\mathcal{R}\langle\partial, \int, \mathrm{E}, \mathcal{S}\rangle$ corresponds to composition of operators and we denote it by $\cdot$ to distinguish it from the multiplication in $\mathcal{R}$. The identities given in Table 1 directly correspond to the defining properties of the operations of $(\mathcal{R}, \partial, \int, \mathcal{S})$. They also have important consequences listed in Table 2. The identities given by Tables 1 and 2 can be used as a *reduction system* in the following way. If the left hand side of one of these identities appears in an expression of an element in $\mathcal{R}\langle\partial, \int, \mathrm{E}, \mathcal{S}\rangle$, we replace it by the right hand side to obtain a new expression for the same element. Continuing this process as long as possible yields an *irreducible form* of that element.

Theorem 7. Let $(\mathcal{R}, \partial, \int, \mathcal{S})$ be an integro-differential ring with linear substitutions. Then, every element of the ring $\mathcal{R}\langle\partial, \int, \mathrm{E}, \mathcal{S}\rangle$ can be written as a sum of irreducible terms of the form

$$A \cdot \mathrm{E} \cdot \sigma_{a,b} \cdot \partial^j \quad \text{and} \quad A \cdot \mathrm{E} \cdot \sigma_{a,b} \cdot \textstyle\int \cdot B,$$

where $j \in \mathbb{N}_0$, each of $A, B \in \mathcal{R}$, E, and $\sigma_{a,b} \in \mathcal{S}$ may be absent, except that $\mathrm{E} \cdot \sigma_{a,b} \cdot \int$ should not specialize to $\mathrm{E} \cdot \int$.

For the technical details, uniqueness of irreducible forms, and proofs, see (Hossein Poor et al., 2018).

Table 2. Consequences of Operator Identities

$\mathrm{E} \cdot \mathrm{E} = \mathrm{E}$	$\int \cdot A \cdot \mathrm{E} = \int A \cdot \mathrm{E}$
$\mathrm{E} \cdot \int = 0$	$\int \cdot A \cdot \partial = A - \int \cdot \partial A - (\mathrm{E}A)\mathrm{E}$
$\partial \cdot \mathrm{E} = 0$	$\int \cdot A \cdot \int = \int A \cdot \int - \int \cdot \int A$
$\int \cdot \mathrm{E} = \int 1 \cdot \mathrm{E}$	$\int \cdot \sigma_{a,b} = a^{-1}(1 - \mathrm{E}) \cdot \sigma_{a,b} \cdot \int$
$\int \cdot \int = \int 1 \cdot \int - \int \cdot \int 1$	$\int \cdot A \cdot \sigma_{a,b} = a^{-1}(1 - \mathrm{E}) \cdot \sigma_{a,b} \cdot \int \cdot \sigma_{a,b}^{-1}A$

3.3 Symbolic Computations with IDOLS

The algebraization of the operators introduced above provides a useful tool for computations with them and with the systems they describe. One advantage of this framework over computations with entries and functions is that computations usually are shorter, as can be seen in the examples below. Moreover, symbolic computations allow to compute with operators corresponding to systems of generic sizes. Based on that, one can also find and prove identities that describe properties of systems of generic sizes. Thanks to the irreducible forms, verification of identities of operators is straightforward.

In order to solve operator equations by ansatz, we need to compute with operators that have undetermined coefficients. This can be achieved by considering the free integro-differential ring with linear substitutions generated by the coefficients occurring, which includes the undetermined ones. By Theorem 7, it suffices to include irreducible terms in the ansatz. Then, after computing irreducible forms of both sides of the operator equation, comparing coefficients results in equations for the undetermined coefficients in the ansatz. These equations need to be solved to obtain a solution of the original equations for operators. In other words, the ansatz reduces equations for operators to equations for coefficients. These equations for coefficients are always sufficient and under suitable genericity assumptions, which are always satisfied in the free integro-differential ring with linear substitutions, they are also necessary.

We illustrate the use of integro-differential operators with linear substitutions over a free integro-differential ring with linear substitutions, by computing a right inverse of a first-order differential operator by ansatz. Thereby we recover the formula (and a short proof) for variation of constants for a linear first-order differential system of generic size. We use this right inverse also to derive the general solution of a first-order differential system. In a second example, we show how to derive, via computations with operators, the method of steps for a linear first-order differential time-delay system.

Example 8. Consider the differential system

$$\dot{x}(t) - A_0(t)x(t) = f(t)$$

which corresponds to the differential operator $L := \partial - A_0$. In order to construct an operator that solves the system, we make the irreducible ansatz $H := H_1 \cdot \int \cdot H_2$ for a right inverse of L, with undetermined multiplication operators H_1 and H_2. So, we take the free integro-differential ring with linear substitutions generated by A_0, H_1, H_2. Using the reduction system, we write the product $L \cdot H$ in irreducible form.

$$\begin{aligned} L \cdot H &= (\partial - A_0) \cdot H_1 \cdot \textstyle\int \cdot H_2 \\ &= (H_1 \cdot \partial + \partial H_1) \cdot \textstyle\int \cdot H_2 - A_0 H_1 \cdot \textstyle\int \cdot H_2 \\ &= H_1 H_2 + (\partial H_1 - A_0 H_1) \cdot \textstyle\int \cdot H_2 \end{aligned}$$

Comparing coefficients in $L \cdot H = 1$ yields

$$H_1 H_2 = 1 \quad \text{and} \quad \partial H_1 - A_0 H_1 = 0.$$

For solving these equations, we adjoin an invertible Φ such that $\partial\Phi - A_0\Phi = 0$ and let $H_1 = \Phi$ and $H_2 = \Phi^{-1}$, i.e.

$$H = \Phi \cdot \textstyle\int \cdot \Phi^{-1}.$$

This is exactly the formula $x = Hf$ for a particular solution of $Lx = f$ that is obtained from a fundamental matrix by variation of constants:

$$x(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) f(s)\, ds.$$

Moreover, the equation $Lx = f$ is equivalent to the equation $(H \cdot L)x = Hf$. By the reduction system, we easily find the irreducible form

$$
\begin{aligned}
H \cdot L &= \Phi \cdot \textstyle\int \cdot \Phi^{-1} \cdot (\partial - A_0) \\
&= \Phi \cdot (\Phi^{-1} - \textstyle\int \cdot \partial(\Phi^{-1}) - (\mathrm{E}\Phi^{-1})\mathrm{E}) - \Phi \cdot \textstyle\int \cdot \Phi^{-1} A_0 \\
&= 1 - \Phi \mathrm{E} \Phi^{-1} \cdot \mathrm{E},
\end{aligned}
$$

where we used the identity $\partial(\Phi^{-1}) + \Phi^{-1}A_0 = 0$ obtained in Example 5. Defining the projector $P = \Phi\mathrm{E}\Phi^{-1}\cdot\mathrm{E}$ allows us to write $(H \cdot L)x = Hf$ as $x = Px + Hf$, which yields the general solution obtained by variation of constants:

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)f(s)\,ds.$$

Example 9. Now, consider the DTD system

$$\dot{x}(t) - A_0(t)x(t) - A_1(t)x(t-h) = f(t)$$

corresponding to the operator $R := L + S$ with differential part $L = \partial - A_0$ as in Example 8 and time-delay part $S := -A_1 \cdot \delta$. For solving this system, like above, we first note that the equation $Rx = f$ is equivalent to the equation $(H \cdot R)x = Hf$ similar to above. Based on $H \cdot R = 1 - G$, where

$$G := P - H \cdot S = \Phi\mathrm{E}\Phi^{-1} \cdot \mathrm{E} + \Phi \cdot \textstyle\int \cdot \Phi^{-1} \cdot A_1 \cdot \delta,$$

we can rewrite $(H \cdot R)x = Hf$ as the recurrence equation

$$x = Gx + Hf.$$

This is the operator interpretation of the method of steps, see e.g. (Hale and Verduyn Lunel, 1993):

$$
\begin{aligned}
x(t) = \Phi(t)\Big(&\Phi^{-1}(t_0)x(t_0) \\
&+ \int_{t_0}^{t} \Phi^{-1}(s)\big(f(s) + A_1(s)x(s-h)\big)ds\Big).
\end{aligned}
$$

3.4 Rectangular Coefficients

Working with operators that have different domains and codomains (e.g., they have rectangular matrices as their coefficients), not all operators can be added or multiplied together. However, in the ring $\mathcal{R}\langle\partial, \int, \mathrm{E}, \mathcal{S}\rangle$ constructed above, there is no restriction on addition and multiplication. Still, when applying the reduction system given in Tables 1 and 2, valid expressions in $\mathcal{R}\langle\partial, \int, \mathrm{E}, \mathcal{S}\rangle$ are transformed into valid expressions, i.e. they can again be interpreted as actual operators with domain and codomain. This can be seen by observing that in each reduction rule the right hand side is automatically an operator with the same domain and codomain whenever the left hand side is. For that, it is important to note that, for every $A \in \mathcal{R}$ that can be interpreted as an operator $A : \mathcal{F}^m \longrightarrow \mathcal{F}^n$, the operations $\partial, \int, \mathrm{E}$ and all $\sigma_{a,b} \in \mathcal{S}$ yield operators ∂A, $\int A$, $\mathrm{E}A$, and $\sigma_{a,b}A$ with the same domain and the same codomain. The generators $\partial, \int, \mathrm{E}$ and all $\sigma_{a,b} \in \mathcal{S}$ of $\mathcal{R}\langle\partial, \int, \mathrm{E}, \mathcal{S}\rangle$ are interpreted as operators from any $\mathcal{F}^n$ to itself. For example, we now explicitly check that the Leibniz rule transforms valid expressions into valid expressions. If some $A \in \mathcal{R}$ can be interpreted as an operator $A : \mathcal{F}^m \longrightarrow \mathcal{F}^n$, then in $\partial \cdot A$ the derivation is interpreted as an operator $\partial : \mathcal{F}^n \longrightarrow \mathcal{F}^n$ and the Leibniz rule

$$\partial \cdot A = A \cdot \partial + \partial A$$

yields $A \cdot \partial$, where we interpret the symbol ∂ as an operator $\partial : \mathcal{F}^m \longrightarrow \mathcal{F}^m$, and ∂A which both map from $\mathcal{F}^m$ to $\mathcal{F}^n$.

The same issue arises with computations in $\mathcal{R}$. By applying the operators on the left and right hand sides of the identities in Tables 1 and 2 to elements of $\mathcal{R}$, we obtain identities in $\mathcal{R}$. Analogously to above, we can check that in each of those identities in $\mathcal{R}$, whenever one term can be interpreted as operator from some $\mathcal{F}^m$ to some $\mathcal{F}^n$ also the other terms can be interpreted as having the same domain and codomain.

Example 10. Consider the rectangular differential system

$$A_1(t)\dot{x}(t) - A_0(t)x(t) = f(t)$$

corresponding to the operator $L := A_1 \cdot \partial - A_0$. Like before, we make the irreducible ansatz $H := H_1 \cdot \int \cdot H_2$ for a right inverse of L, with undetermined multiplication operators H_1 and H_2. Then, using the reduction system, we write the product $L \cdot H$ in irreducible form.

$$
\begin{aligned}
L \cdot H &= (A_1 \cdot \partial - A_0) \cdot H_1 \cdot \textstyle\int \cdot H_2 \\
&= (A_1H_1 \cdot \partial + A_1\partial H_1) \cdot \textstyle\int \cdot H_2 - A_0H_1 \cdot \textstyle\int \cdot H_2 \\
&= A_1H_1H_2 + (A_1\partial H_1 - A_0H_1) \cdot \textstyle\int \cdot H_2
\end{aligned}
$$

Comparing coefficients in $L \cdot H = 1$ yields

$$A_1H_1H_2 = 1 \quad \text{and} \quad A_1\partial H_1 - A_0H_1 = 0.$$

To solve these equations, we adjoin Θ and $\tilde{\Theta}$ s.t. $A_1\Theta\tilde{\Theta} = 1$ and $A_1\partial\Theta - A_0\Theta = 0$ and we let $H_1 = \Theta$ and $H_2 = \tilde{\Theta}$.

3.5 Implementation in Software

In order to assist computations with operators in the ring of IDOLS, we provide a `Mathematica` package that implements the arithmetics of this ring. The main functionality is to compute irreducible forms via the reduction system and to extract coefficients. Arithmetics for the free integro-differential ring with linear substitutions (i.e., computations with coefficients) is also implemented and there is support for block matrices. The examples discussed so far are sufficiently short and simple to be computed by hand, still we use them below to briefly explain the use of the package. A bigger application will be presented in Section 4, where finding and proving Artstein's transformation is supported by our package.

When solving operator equations by ansatz, computation of the irreducible form and coefficient comparison can be done using the package. For instance, in order to compute a right inverse $H = H_1 \cdot \int \cdot H_2$ of a differential operator $L = \partial - A_0$ as above, the user first has to declare the coefficients A_0, H_1, H_2. Then, by the package, the equations for the coefficients are obtained. For introducing a solution, the user can declare the new coefficient Φ with the property $\partial\Phi = A_0\Phi$. When using Φ^{-1}, the implied property $\partial(\Phi^{-1}) = -\Phi^{-1}A_0$ is automatically inferred by the package. With these definitions the package returns the irreducible form $1 - \Phi\mathrm{E}\Phi^{-1} \cdot \mathrm{E}$ of $H \cdot L$.

For proving operator identities, irreducible forms can be computed and compared by the software. For instance, the package can be used to prove $L \cdot H = 1$ with $H = \Phi \cdot \int \cdot \Phi^{-1}$ and L as defined above.

4. RECOVERING ARTSTEIN'S TRANSFORMATION

In this section, following the work of Quadrat (2015), we show how our package can be used effectively to recover and prove Artstein's transformation for DTD control systems of the form

$$\dot{x}(t) = A(t)x(t) + B_0(t)u(t) + B_1(t)u(t-h). \tag{7}$$

To apply the algebraic framework introduced in Section 3, we write this control system as a differential time-delay system where coefficient matrices have block structure:

$$(I_n\ 0)\begin{pmatrix}\dot{x}(t)\\ \dot{u}(t)\end{pmatrix} = \\ (A(t)\ B_0(t))\begin{pmatrix}x(t)\\ u(t)\end{pmatrix} + (0\ B_1(t))\begin{pmatrix}x(t-h)\\ u(t-h)\end{pmatrix}. \quad (8)$$

We show how to use our software to find by ansatz (and then prove) a transformation from the DTD system (7) to the differential system

$$\dot{z}(t) = E(t)z(t) + F(t)v(t), \quad (9)$$

considered as a differential system with block structure

$$(I_n\ 0)\begin{pmatrix}\dot{z}(t)\\ \dot{v}(t)\end{pmatrix} = (E(t)\ F(t))\begin{pmatrix}z(t)\\ v(t)\end{pmatrix}. \quad (10)$$

The systems (8) and (10) correspond to the operators

$$R' = R'_0 \cdot \partial + R'_1 + R'_2 \cdot \delta, \quad (11)$$

$$R = R_0 \cdot \partial + R_1, \quad (12)$$

where

$$R'_0 = (I_n\ 0), \quad R'_1 = (-A\ -B_0), \quad R'_2 = (0\ -B_1),$$
$$R_0 = (I_n\ 0), \quad R_1 = (-E\ -F).$$

Based on (2), our goal is to find P and Q such that

$$R \cdot P = Q \cdot R'. \quad (13)$$

We choose $Q = Q_0$ where Q_0 is a multiplication operator, and consider the following ansatz for the operator P

$$P = P_0 \cdot \delta \cdot \textstyle\int \cdot P_1 + P_2 \cdot \int \cdot P_3 + P_4 \cdot \delta + P_5, \quad (14)$$

where multiplication operators $P_0, P_1, P_2, P_3, P_4, P_5$ have undetermined blocks $P_{11}, P_{22}, a_0, a_1, a_2, a_3, a_4, a_5$:

$$P_0 = \begin{pmatrix}a_0\\ 0\end{pmatrix}, \ P_1 = (0\ a_1), \ P_2 = \begin{pmatrix}a_2\\ 0\end{pmatrix}, P_3 = (0\ a_3),$$
$$P_4 = \begin{pmatrix}0 & a_4\\ 0 & 0\end{pmatrix}, \ P_5 = \begin{pmatrix}P_{11} & a_5\\ 0 & P_{22}\end{pmatrix}.$$

With this ansatz, our software computes the conditions for the coefficients in order to have (13). First, the irreducible forms for the left and the right hand sides of (13) are computed. Then, by coefficient comparison, the following conditions are obtained for the blocks $a_0, a_1, a_2, a_3, a_4, a_5, P_{11}, P_{22}, Q_0$.

$$\partial a_0 - E a_0 = 0 \quad (15)$$
$$\partial a_2 - E a_2 = 0 \quad (16)$$
$$P_{11} = Q_0 \quad (17)$$
$$a_4 = 0 \quad (18)$$
$$a_5 = 0 \quad (19)$$
$$a_0 \delta a_1 + \partial a_4 - E a_4 = -Q_0 B_1 \quad (20)$$
$$\partial P_{11} - E P_{11} = -Q_0 A \quad (21)$$
$$\partial a_5 + a_2 a_3 - E a_5 - F P_{22} = -Q_0 B_0 \quad (22)$$

For solving these equations, following (Quadrat, 2015), we set $a_4 = a_5 = 0$ and we let P_{11} be such that (21) holds. Furthermore, we set $Q_0 = P_{11}$ and we let Φ be invertible such that

$$\partial \Phi = E\Phi.$$

Then, for arbitrary constants c_0 and c_2, we assume that

$$a_0 = \Phi c_0 \quad \text{and} \quad a_2 = \Phi c_2.$$

This solves six of the above equations. The remaining equations can now be written as

$$c_0 a_1 = -\delta^{-1}\Phi^{-1}P_{11}B_1,$$
$$c_2 a_3 = \Phi^{-1}(FP_{22} - P_{11}B_0),$$

and we assume that c_0, c_2, a_1, a_3 are such that they satisfy these equations. After entering these assumptions, our package verifies that all conditions (15) through (22) are satisfied, also (13) can be verified directly. With these assumptions, (14) can be rewritten as

$$P = -\begin{pmatrix}\Phi\\ 0\end{pmatrix} \cdot \delta \cdot \textstyle\int \cdot \left(0\ \delta^{-1}\Phi^{-1}P_{11}B_1\right) \\ + \begin{pmatrix}\Phi\\ 0\end{pmatrix} \cdot \textstyle\int \cdot \left(0\ \Phi^{-1}(FP_{22} - P_{11}B_0)\right) + \begin{pmatrix}P_{11} & 0\\ 0 & P_{22}\end{pmatrix}.$$

In other words, we obtain the invertible transformation

$$\begin{pmatrix}z(t)\\ v(t)\end{pmatrix} = \begin{pmatrix}P_{11}(t)x(t)\\ P_{22}(t)u(t)\end{pmatrix} + \begin{pmatrix}\Phi(t)\\ 0\end{pmatrix}\int_{t_0}^{t}\Phi^{-1}(s)T_2(s)u(s)\,ds \\ - \begin{pmatrix}\Phi(t)\\ 0\end{pmatrix}\int_{t_0}^{t-h}\Phi^{-1}(s+h)P_{11}(s+h)B_1(s+h)u(s)\,ds,$$

where $T_2(t) := F(t)P_{22}(t) - P_{11}(t)B_0(t)$, as in Theorem 5 of (Quadrat, 2015).

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