On Feedback Stabilization of Neutral Time Delay Systems with Infinitely Many Unstable Poles

Suat Gumussoy * **Hitay Özbay** **

* *MathWorks, 3 Apple Hill Drive, Natick, MA 01760, USA (e-mail: Suat.Gumussoy@mathworks.com)*
** *Department of Electrical and Electronics Engineering, Bilkent University, 06800 Ankara, Turkey, (e-mail: hitay@bilkent.edu.tr)*

Abstract: It is shown that strictly proper neutral time delay systems having at least one asymptotic pole chain converging to a vertical line $Re(s) = \sigma > 0$ cannot be stabilized by a proper controller. A special coprime factorization can be obtained for such systems with infinitely many unstable poles, provided they are bi-proper (i.e. proper, but not strictly proper). From this factorization, stabilizing feedback controllers are obtained. Necessarily, all stabilizing controllers for this type of plants are also bi-proper.

Keywords: Time delay, Factorization methods, Feedback stabilization, Unstable systems, Coprime factorization

1. INTRODUCTION

Neutral systems appear in various applications where there is an internal delay in the highest order derivative in the differential equation describing the dynamical behavior of the underlying physical system, see e.g. Michiels and Niculescu (2007). Such a delay coupling in the ODEs (as well as PDEs) result in an infinite dimensional system whose feedback stabilization is an interesting subject, see e.g. Burns et al. (2013), Nguyen and Bonnet (2015), Rabah et al. (2008) and their references. In particular, Byrnes et al. (1984), Burns et al. (2013), and Rabah et al. (2008) investigate linear systems represented by state-space models and propose methods for stabilization by state feedback. In the present work, we consider single-input-single-output (SISO) linear time invariant neutral time delay systems represented by their transfer functions, i.e., an input-output approach is adopted.

One of the main difficulties associated with this class of systems is that they may have infinitely many unstable poles. There are many works where pole locations are examined. For example, Rabah et al. (2005) performs a spectrum analysis for a neutral system given in state-space, that gives estimates of the pole locations. Recently, Nguyen et al. (2016) gives a condition for the H_∞-stability of a class of neutral fractional delay systems, where all poles are in the open left half plane but there is at least one asymptotic chain of poles approaching the imaginary axis, denoted by $\mathbb{I} = \{s \in \mathbb{C} \ : \ Re(s) = 0\}$. As briefly summarized in the next section, for neutral systems with commensurate delays, there exist simple tests to check if there is a chain of poles asymptotic to a vertical axis in the open right half plane, see e.g. Loiseau et al. (2002), Gumussoy (2012) and their references.

Earlier in Gumussoy and Ozbay (2004) stabilizing feedback controllers are obtained for a class of neutral systems having infinitely many unstable poles. On the other hand, in Loiseau et al. (2002) it has been shown that having finitely many unstable poles is a necessary condition for the existence of a stabilizing feedback controller for many interesting neutral delay systems. At this point we would like to quote the following relevant discussion from Loiseau et al. (2002):

> "According to Byrnes *et al.* (1984), O'Connor & Tarn (1983), and Pandolfi (1976) if a system is not formally stable, then the stabilizing compensator should include some derivative action."
>
> "Byrnes *et al.* (1984), ...notice that finding general necessary and sufficient conditions for the stabilization of neutral-type time-delay systems is an open question."
>
> "We establish that a time-delay system of neutral-type that is not formally stable cannot be stabilized by any state feedback ..."

Note that in the above discussion "formally stable" is equivalent to having finitely many unstable poles, see Loiseau et al. (2002) and their reference to Pontryagin.

An implicit assumption in Gumussoy and Ozbay (2004) was that the plant is bi-proper (i.e. proper, but not strictly proper). Moreover, the necessary condition established in Loiseau et al. (2002) is valid for the state-space models (their transfer functions are strictly proper). Requiring a derivative action to stabilize such systems with infinitely many unstable poles is equivalent to imposing an improper controller. Therefore, it seems that, for a SISO neutral time delay system which has infinitely many unstable poles, having a *bi-proper transfer function* is a necessary condition for the existence of a *proper stabilizing controller*. Yet, we could not find this result explicitly stated in the literature (there are some relevant results in Nguyen and Bonnet (2014) and Partington and Bonnet (2004), where some special cases are considered, we will discuss these below). In Section 2 we prove this fact. In Section 3 we recall stabilizing controller design using special factorizations. Concluding remarks are made in Section 4.

2. MAIN RESULT

2.1 Problem Definition

In this paper we consider plants of the form

$$P(s) = \frac{r(s)}{q(s)} \tag{1}$$

where $r(s)$ and $q(s)$ are quasi-polynomials defined by polynomials $r_i(s)$, $i = 1, \ldots, n$, $q_k(s)$, $k = 1, \ldots, m$, and delays $0 \le \tau_1 < \cdots < \tau_n$, $0 \le h_1 < \cdots < h_m$,

$$r(s) = \textstyle\sum_{i=1}^{n} r_i(s) e^{-\tau_i s} \ ,$$

$$q(s) = \textstyle\sum_{k=1}^{m} q_k(s) e^{-h_k s} \ .$$

Assumptions. The following assumptions on the plant will be in effect:

- *A1* Time delays are commensurate: there exists $\tau_o > 0$ and $h_o > 0$ such that $\tau_i = (i-1)\tau_o$, $i = 1, \ldots, n$, and $h_k = (k-1)h_o$, $k = 1, \ldots, m$.
- *A2* $q(s)$ is a neutral quasi-polynomial: deg $q_1 \ge$ deg q_k, $k = 1, \ldots, m$ and there exists at least one $k \in \{2, \ldots, m\}$ such that deg $q_1 =$ deg q_k
- *A3* $q(s)$ has infinitely many roots in a half plane $\mathbb{C}_\sigma$ defined as $\mathbb{C}_\sigma = \{s \ : \ Re(s) \ge \sigma\}$, with $\sigma > 0$; and there exists $\epsilon > 0$ such that $q(s)$ has finitely many roots in the strip $\mathbb{C}_{-\epsilon} \setminus \mathbb{C}_{\epsilon}$, (i.e. ϵ-neighborhood of $\mathbb{I}$).
- *A4* $r(s)$ can be a retarded or neutral type: deg $r_1 \ge$ deg r_i for all $i = 1, \ldots n$; but if $r(s)$ is a neutral quasi-polynomial, then it has finitely many roots in $\mathbb{C}_{-\epsilon}$ for some $\epsilon > 0$.
- *A5* None of the roots of $r(s)$ in $\mathbb{C}_{-\epsilon}$ coincide with the roots of $q(s)$.
- *A6* $P(s)$ is a proper transfer function: deg $q_1 \ge$ deg r_1.

Note that, the first assumption implies $\tau_1 = 0$, which rules out feedforward input-output delay in the plant transfer function. Then, for causality (to avoid time advance) we also need $h_1 = 0$.

Under the above definitions and assumptions *A1*—*A6*, a factorization $P = N/D$ can be obtained by setting

$$N(s) := r(s)/(s+a)^{\ell_r} \ , \quad D(s) = q(s)/(s+b)^{\ell_q}$$

where $a, b > 0$ are arbitrary and $\ell_r =$ deg r_1, $\ell_q =$ deg q_1. Note that both N and D are in $\mathcal{H}_\infty$. The main question is this: are N and D strongly coprime in $\mathcal{H}_\infty$? If so, how can we find $X, Y \in \mathcal{H}_\infty$ such that $NX + DY = 1$? When the answers are positive, the plant is $\mathcal{H}_\infty$-stabilizable, and all stabilizing controllers are obtained from Smith (1989). Below, we will see that such a stabilizing controller exists *only if* $\ell_r = \ell_q$, i.e., a necessary condition is to have a plant transfer function which is bi-proper (proper, but not strictly proper). Moreover when this necessary condition is satisfied we will show that it is possible to find the required X, Y and obtain stabilizing controllers.

In passing, we want to mention that the plants of the form (1) can be obtained from "state-space" dynamical models, such as

$$\dot{x}(t) - E\dot{x}(t-h) = A_0 x(t) + A_1 x(t-h) + Bu(t) \tag{2}$$

$$y(t) = C_0 x(t) + C_1 x(t-\tau) + Du(t) \tag{3}$$

where $x(t) \in \mathbb{R}^{n_x}$, $h > 0$, $\tau > 0$, D scalar, and the matrices E, A_0, A_1 are $n_x \times n_x$, C_0, C_1 are $1 \times n_x$, and B is $n_x \times 1$. Taking the Laplace transforms of (2) and (3) and eliminating $X(s)$ we obtain $P(s) = Y(s)/U(s)$ which is of the form (1). Of course, it is possible to add extra commensurate delay terms to the above model and have a transfer function similar to $P(s)$.

Definition 1. Let $q(s)$ be as above. The function

$$a^q(z) = \sum_{k=1}^{m} a_k z^{k-1}$$

is called the *asymptotic polynomial* of $q(s)$ where

$$a_k = \lim_{s \to \infty} \tfrac{q_k(s)}{q_1(s)}, \text{ for } k = 1, \ldots, m.$$

We recall the following test from Gumussoy (2012).

Lemma 2. The quasi-polynomial $q(s)$ as defined above has finitely many roots in $\mathbb{C}_+$ if and only if its asymptotic polynomial $a^q(z)$ has all its roots outside the unit circle.

For example, when $m = 2$ the asymptotic polynomial of $q(s)$ is of the form $a^q(z) = (1 + a_2 z)$ where $a_2 = \lim_{s \to \infty} \frac{q_2(s)}{q_1(s)}$. Since q is assumed to be neutral, we have deg $q_2 =$ deg q_1 and hence $a_2 \neq 0$. According to Lemma 2, q has has finitely many roots in $\mathbb{C}_+$ if and only if $|a_2| < 1$, which is consistent with Proposition 2.1 of Partington and Bonnet (2004), where the notation $\alpha = a_2^{-1}$ is used. Also, we should mention that for systems represented by (2) having finitely many unstable poles is equivalent to having $|z_k| > 1 \ \forall \ k$, where $z_k \in \mathbb{C}$ is a root of $\det(I - zE) = 0$. This property is the special case of the "formal stability", see Proposition 2.1.4 of Byrnes et al. (1984). It is an interesting exercise to show that when

$$\left(C_0 + C_1 e^{-\tau s}\right) \left(s(I - Ee^{-hs}) - A_0 - A_1 e^{-hs}\right)^{-1} B + D \\ = r(s)/q(s)$$

we have $q(s) = \det\left(s(I - Ee^{-hs}) - A_0 - A_1 e^{-hs}\right)$ and its asymptotic polynomial is $a^q(z) = \det(I - zE)$.

Many prior works, for systems described by (2), have dealt with the question of finding stabilizing feedback laws of the form

$$u(t) = -\sum_{k=0}^{n_f} F_k x(t - kh) \tag{4}$$

where $F_0, \ldots, F_{n_f}$ are appropriate size matrices. It was shown by Loiseau et al. (2002) that systems of the form (2) are not BIBO stabilizable by any state feedback (4) if they are not "formally stable". See the discussion in Loiseau et al. (2002) for earlier related results.

For plants in the form (1), with the special case $n = 1$ and $m = 2$ a similar result has been obtained in Partington and Bonnet (2004): if $|\alpha| \le 1$ and deg $r_1 =$ deg $q_1 - 1$ then P is not BIBO stabilizable, nor $\mathcal{H}_\infty$-stabilizable by finite dimensional controllers. We should also point out that for the case $r(s) = 1$ and q_k's are polynomials in a fractional power of s, Nguyen and Bonnet (2014) has proven that if the plant has infinitely many unstable poles then it cannot be stabilized by the class of rational fractional controllers of commensurate order. Another earlier related result appears in Yamamoto and Hara (1988), where it has been proven that when a repetitive controller $(1-e^{-Ls})^{-1}$, $L>0$, appears in the open loop transfer function, then it is necessary to have a bi-proper open loop transfer function for stability of the feedback system. Such a system does not satisfy our assumptions *A1–A6*, because it has infinitely many poles on $\mathbb{I}$. But for stabilization purposes one can do an axis shift and try to stabilize the feedback system whose open loop transfer function contains a factor in the form $(1-e^{-L(s-\epsilon)})^{-1}$, $\epsilon>0$, which fits our framework. This approach is useful in placing the closed loop system poles to the left of the line $Re(s)=-\epsilon$.

All the papers mentioned in the above discussion (and many other earlier papers in their references) show that it is impossible to move infinitely many unstable poles to the left half plane by a proper controller when the plant bandwidth is finite. We will formally generalize this fact in the next section.

2.2 A necessary condition for feedback stabilization

Let (C,P) be a feedback system formed by a proper controller $C(s)$ and a proper plant $P(s)$. Then, the closed loop system is stable if $S=(1+PC)^{-1}, CS, PS \in \mathcal{H}_\infty$. This implies that the feedback system is unstable if the characteristic function, $1+G(s)$, has a zero in $\mathbb{C}_+$, where $G(s)=P(s)C(s)$.

Proposition 3. Let $C(s)$ be a proper controller for a proper plant $P(s)$. Assume that there is no unstable pole-zero cancellation in the product $G(s)=P(s)C(s)$, and that G does not have any poles on $\mathbb{I}$; moreover, there exists a sufficiently small $\epsilon>0$ such that G has finitely many poles in the ϵ-neighborhood of $\mathbb{I}$. Further assume that, for any finite $\rho>0$, G has finitely many poles in the closed semi-disk

$$\mathbb{D}_\rho := \{ s=\rho e^{j\theta} \ : \ |s|\le\rho \ , \ \theta\in[-\frac{\pi}{2} \ , \ \frac{\pi}{2}] \ \}.$$

If the following two conditions hold, then the feedback system is unstable:

(i) there exists $\omega_*>0$ such that $|G(j\omega)|<1$ for all $\omega>\omega_*$ and there exists $\rho_*>0$ and a strictly increasing sequence $\rho_n>\rho_*$, $n=1,2,\dots$, with $(\rho_{n+1}-\rho_n)>\delta$ for some $\delta>0$ such that $|G(\rho_n e^{j\theta})|<1$ for all $n=1,2,...$ and all $\theta\in[-\frac{\pi}{2} \ , \ \frac{\pi}{2}]$,
(ii) the number of poles of G inside $\mathbb{D}_{\rho_n}$ increases without a bound as $n\to\infty$.

Proof. Given $\rho=\rho_n>\rho_*$, let us define a positive contour $\overrightarrow{\Gamma}_\rho$ encircling $\mathbb{D}_\rho$, by going over the path $s=j\omega$, ω increasing from $-\rho$ to $+\rho$, and then $s=\rho e^{j\theta}$, θ decreasing from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$. The condition (i) implies that for $\rho=\rho_n$ the transfer function $G=PC$ does not have any poles on $\overrightarrow{\Gamma}_\rho$. According to Cauchy's argument principle, see e.g. Ablowitz and Fokas (2003), the number of zeros, n_z, of the function $1+G(s)$ inside $\overrightarrow{\Gamma}_\rho$ is equal to $(n_p+n_{\circlearrowright})$, where n_p is the number of poles of G inside $\overrightarrow{\Gamma}_\rho$ and $n_{\circlearrowright}$ is the number of clockwise encirclements of -1 by the closed path $\overrightarrow{\Gamma}_G := G(\overrightarrow{\Gamma}_\rho)$. Clearly, a necessary condition for feedback system stability is to have $n_z=0$ for any given $\rho=\rho_n$. That means the curve $\overrightarrow{\Gamma}_G$ must encircle -1 exactly n_p times in the counter clockwise direction. On the other hand, since G satisfies (i), for any given small number $\varepsilon\in(0 \ , \ 1)$, there exists N_ε such that $\rho_n>\omega_*$, and $|G(\rho_n e^{j\theta})|<(1-\varepsilon)$ for all $n>N_\varepsilon$ and $\theta\in[-\frac{\pi}{2} \ , \ \frac{\pi}{2}]$. Therefore, $n_{\circlearrowright}$ is invariant for all n larger than N_ε for any $\varepsilon\in(0 \ , \ 1)$. By condition (ii), the number n_p increases without a bound as $\rho_n\to\infty$. Thus, there is a sufficiently large $\rho=\rho_n$ with $n>N_\varepsilon$, with $0<\varepsilon<1$ for which n_p is strictly greater than $-n_{\circlearrowright}$, which means that $1+G(s)$ has at least one zero inside $\mathbb{D}_\rho$, hence the feedback system is unstable. □

For neutral time delay systems, the condition (i) is satisfied when the plant is strictly proper; the condition (ii) holds when the system has infinitely many poles in $\mathbb{C}_\sigma$ for some $\sigma>0$.

Corollary 4. Let P be a neutral system satisfying *A1–A6*. If P is strictly proper, then it cannot be stabilized by a proper controller.

An important point to note is that Proposition 3 is valid for a large class of infinite dimensional systems, beyond the ones captured by (1) with *A1–A6*. The condition on the imaginary axis poles was to simplify the notation in the proof; if $G(s)$ has finitely many poles on $\mathbb{I}$, then by a slight change in the contour $\overrightarrow{\Gamma}_\rho$, we can arrive at the same conclusion. We have proven that under conditions (i) and (ii) the feedback system has at least one unstable pole. The arguments of the proof further suggests that when a strictly proper G satisfies condition (ii), the feedback system will have infinitely many unstable poles. In other words, it is impossible to move infinitely many unstable poles of P to $\mathbb{C}_-$, by a proper C when P is strictly proper.

Another important point to note is that we have not used the assumption that $\tau_1=0$. In fact, the above result is valid for all plants of the form $e^{-\tau_1 s}P(s)$ where $\tau_1>0$ and P satisfies *A1–A6*; we should mention that the same remark appears in Partington and Bonnet (2004). However, for numerical computation of all stabilizing controllers we will need $\tau_1=0$ in the next section.

3. STABILIZING CONTROLLERS FOR NEUTRAL SYSTEMS WITH INFINITELY MANY UNSTABLE POLES

In the light of Proposition 3, in order to find stabilizing controllers the assumption *A6* must be strengthened to

A6' $P(s)$ is bi-proper, i.e., deg q_1 = deg r_1.

Also, for simplicity of the plant factorizations we modify *A3* as follows:

A3' $q(s)$ has infinitely many roots in $\mathbb{C}_\sigma$, for some $\sigma>0$, and has finitely many roots in $\mathbb{C}\setminus\mathbb{C}_\sigma$, none on $\mathbb{I}$.

In particular, this implies that q has finitely many roots in $\mathbb{C}_-$. The added restriction, (no imaginary axis poles), is not entirely necessary, it can be avoided by an axis shift in the form $\widehat{s}=s+\epsilon$, as long as q has finitely many roots to the left of $Re(s) = -\epsilon$, for some $\epsilon > 0$. We do not go into such technical details here.

Definition 5. Let $q(s)$ be as above. The quasi-polynomial

$$\bar{q}(s) := -q(-s)e^{-h_m s} \tag{5}$$

is called the *conjugate quasi-polynomial* of $q(s)$.

Clearly, by *A3'* the conjugate quasi-polynomial $\bar{q}(s)$ has infinitely many roots to the left of $Re(s) = -\sigma$, finitely many roots in $\mathbb{C}_+$, and none of the roots are on $\mathbb{I}$.

We now recall the following factorization from Gumussoy (2012). First write $P(s)$ as

$$P(s) = \frac{r(s)}{\bar{q}(s)} \frac{\bar{q}(s)}{q(s)}.$$

Then note that by assumption *A3'* it is possible to find a finite Blaschke function $B_{\bar{q}}(s)$ such that

$$D(s) := \frac{q(s)}{\bar{q}(s)} B_{\bar{q}}(s) \in \mathcal{H}_\infty \tag{6}$$

is inner. The zeros of $B_{\bar{q}}$ are the roots of $\bar{q}(s)$ in $\mathbb{C}_+$, having the same multiplicities. Similarly, by assumption *A4*, we can find a finite Blaschke product $B_r(s)$ such that

$$N_o(s) := \frac{r(s)}{\bar{q}(s)} \frac{B_{\bar{q}}(s)}{B_r(s)} \in \mathcal{H}_\infty. \tag{7}$$

is outer. Moreover, by the assumption *A6'* we have that $N_o^{-1} \in \mathcal{H}_\infty$. Now we define $N(s) := B_r(s)N_o(s)$, which is an inner-outer factorization for $N \in \mathcal{H}_\infty$. Then, $P = N/D$ is a strongly coprime factorization for which it is possible to find $X, Y \in \mathcal{H}_\infty$ satisfying the Bezout equation

$$N(s)X(s) + D(s)Y(s) = 1 \ .$$

The computations for X and Y are done as outlined in Ozbay et al. (1990): let $X = X_r N_o^{-1}$, then $X_r \in \mathcal{H}_\infty$ must satisfy

$$X_r(s) = \frac{1 - D(s)Y(s)}{B_r(s)}. \tag{8}$$

Since B_r is finite dimensional, it is always possible to find a rational $Y \in \mathcal{H}_\infty$ which makes $X_r \in \mathcal{H}_\infty$. Just to illustrate this point assume that the zeros of $B_r(s)$ are distinct and denoted by $\beta_1, \ldots, \beta_\ell \in \mathbb{C}_+$. Then $Y \in \mathcal{H}_\infty$ must satisfy interpolation conditions $Y(\beta_i) = 1/D(\beta_i)$. It is possible to construct an $(\ell - 1)$ order $Y \in \mathcal{H}_\infty$ satisfying these conditions. Then, by Smith (1989) all stabilizing controllers for P are given by

$$C = \frac{X_r N_o^{-1} + DQ}{Y - B_r N_o Q} = \left(\frac{X_r + DQ_o}{Y - B_r Q_o} \right) \ N_o^{-1} \tag{9}$$

where $Q = N_o Q_o$ with the free parameter $Q_o \in \mathcal{H}_\infty$, subject to the restriction that $Q_o(\infty) \neq Y(\infty)B_r(\infty)^{-1}$ (the added restriction is needed because the controller must be proper). Obviously, the invertible outer part of the plant does not play a major role in the stabilization problem: we just need to find a stabilizing controller $\widehat{C} = X_r/Y$ for $\widehat{P} = B_r/D$, then $C = \widehat{C}N_o^{-1}$ is stabilizing $P = \widehat{P}N_o$. Under the controller (9) the closed loop transfer functions are

$$\begin{aligned} S &= D(Y - B_r Q_o) \ , \\ CS &= D(X_r + DQ_o)N_o^{-1} \ , \\ PS &= B_r(Y - B_r Q_o)N_o. \end{aligned}$$

Example 1. The following plant satisfies all the assumptions: $P(s) = r(s)/q(s)$ with

$$r(s) = 3\ (s - 2e^{-0.1s})\ , \quad q(s) = (s + 3 + 2(s-1)e^{-0.4s}).$$

The quasi-polynomial r is a retarded-type and has only one unstable pole, at $s = \beta \approx 1.68916$. The quasi-polynomial $q(s)$ is neutral-type; the root of its asymptotic polynomial $a^q(z) = 1 + 2z$ is inside the unit circle. Hence $q(s)$ has infinitely many roots in $\mathbb{C}_+$. Using available numerical methods, such as `QPmR.m` of Vyhlidal and Zitek (2009, 2014), and YALTA of Avanessoff et al. (2013, 2015) we can check that the conjugate quasi-polynomial $\bar{q}(s) = 2(s+1) + (s-3)e^{-0.4s}$ has only one pole in $\mathbb{C}_+$, at $s = \varrho \approx 0.247$. Then, the factorization is in the form $P = B_r N_o/D$ where

$$\begin{aligned} B_r(s) &= \frac{s - \beta}{s + \beta} \\ D(s) &= \frac{(s + 3 + 2(s-1)e^{-0.4s})\ (s - \varrho)}{(2(s+1) + (s-3)e^{-0.4s})\ (s + \varrho)} \\ N_o(s) &= \frac{3(s - 2e^{-0.1s})(s + \beta)(s - \varrho)}{(2(s+1) + (s-3)e^{-0.4s})(s - \beta)(s + \varrho)}. \end{aligned}$$

Note that $D \in \mathcal{H}_\infty$ is inner with infinitely many zeros in $\mathbb{C}_+$. The outer factor N_o is invertible in $\mathcal{H}_\infty$, in particular, it is bi-proper and it does not have a zero at ϱ and does not have a pole at β. The above representations of D and N_o seem to be very sensitive to errors in the computation of the roots β and ϱ. On the other hand, these transfer functions can be implemented in a reliable manner, using stable "finite-impulse-response (FIR)" terms as illustrated in Gumussoy (2012) for certain factors of $\mathcal{H}_\infty$ optimal controllers. Returning back to the stabilization problem, we need to find $Y \in \mathcal{H}_\infty$ such that $Y(\beta) = 1/D(\beta)$. The simplest solution is the constant function, $Y(s) = 1/D(\beta)$. Then,

$$X_r(s) = \left(1 - \frac{D(s)}{D(\beta)} \right) \ \left(\frac{s + \beta}{s - \beta} \right) \in \mathcal{H}_\infty$$

and hence all stabilizing controllers are obtained from the parameterization

$$\begin{aligned} C &= \left(\frac{X_r + DQ_o}{Y - B_r Q_o} \right) \ N_o^{-1} \\ &= \left(\frac{(D(\beta) - D(s)) \left(\frac{s+\beta}{s-\beta} \right) + D(s)Q_1(s)}{1 - \left(\frac{s-\beta}{s+\beta} \right) Q_1(s)} \right) \ N_o^{-1} \end{aligned}$$

where $Q_1(s) = D(\beta)Q_o(s) \in \mathcal{H}_\infty$ is the free parameter, subject to $Q_1(\infty) \neq 1$. A particular stabilizing controller is obtained by putting $Q_1 = 0$,

$$C_0(s) = \Big(D(\beta) - D(s) \Big) \left(\frac{s + \beta}{s - \beta} \right) \ N_o^{-1}(s).$$

Example 2. The purpose of this example is to illustrate that the approach taken here extends to a certain *advanced-type* time delay systems. Consider the bi-proper transfer function

$$P(s) = \frac{(s-1)\ (s+1)}{q(s)}$$

where $q(s)$ is an advanced-type quasi-polynomial in the form

$$q(s) = c_0 s + (c_0 c_1 - s^2)e^{-hs} - c_1 s e^{-2hs}$$

with $h > 0$ and $c_0, c_1 \in [0, \frac{\pi}{2h})$. In this case $q(s)$ has infinitely many roots in $\mathbb{C}_+$ and infinitely many roots in $\mathbb{C}_-$ (the assumption *A3'* is violated). To see this, note that $q(s)$ can be factored as

$$q(s) = (c_0 - se^{-hs})\ (s + c_1 e^{-hs})$$

the first term has all its infinitely many roots in $\mathbb{C}_+$ and the second term has all its infinitely many roots in $\mathbb{C}_-$, as long as c_0 and c_1 are sufficiently small. In the light of this observation, using the conjugate of the first term, a factorization in the form $P = B_r N_o D^{-1}$ is obtained as

$$\begin{aligned} B_r(s) &= \frac{(s-1)}{(s+1)} \\ N_o(s) &= \frac{(s+1)^2}{(s + c_0 e^{-hs})(s + c_1 e^{-hs})} \\ D(s) &= \frac{c_0 - se^{-hs}}{s + c_0 e^{-hs}}\ . \end{aligned}$$

Transfer functions of the form $D(s)$, where the numerator is advanced type quasi-polynomial, appear in the class of *pseudo-rational input-output maps*, see Yamamoto (1988). Once we have the factorization $P = B_r N_o D^{-1}$, the stabilizing controllers can be obtained as in (9). Note that, compared to the plant given in Example 1, this system does not require any root computation for quasi-polynomials.

4. CONCLUSION

We have seen that if the plant contains infinitely many unstable poles in a positive half plane $Re(s) \geq \sigma > 0$, then for feedback system stability the open loop transfer function $G(s) = P(s)C(s)$ cannot be strictly proper. In particular, this means that a *proper* (respectively strictly proper) controller cannot stabilize such a *strictly proper* (respectively proper) plant. From the discussion of Section II it is clear that the main result is applicable to wide range of plants beyond the most natural case, neutral time delay systems having infinitely many unstable poles.

For a certain subclass of bi-proper plants having infinitely many unstable poles we have given coprime factorizations, and a method to compute stabilizing controllers. This method relies on the assumption that the inner part of the numerator $B_r(s)$ is a finite Blaschke product. Finding numerically reliable methods for computing stabilizing controllers when $\tau_1 > 0$, or $r(s)$ is a neutral system having infinitely many roots in $\mathbb{C}_+$, is still an open question. Because in this case both X and Y in the Bezout equation need to be infinite dimensional.

In the above approach we need to find the roots of $r(s)$ in $\mathbb{C}_+$ and the roots of $q(s)$ in $\mathbb{C}_-$ (these are assumed to be finitely many). At this point, establishing conditions for stability robustness of the feedback system, under numerical computation errors for these roots, is another open issue. However, this should not be difficult once the plant and controller coprime factors are expressed as $\mathcal{H}_\infty$ functions with bounded $\mathcal{H}_\infty$ perturbations (e.g. using FIR representations), there is a well known robust stability test for such uncertain systems, see Georgiou and Smith (1990).

ACKNOWLEDGEMENTS

The authors would like to thank Catherine Bonnet and Yutaka Yamamoto for fruitful discussions on the subject.

REFERENCES

M. J. Ablowitz, and A. S. Fokas. *Complex Variables*, 2nd Ed.. Cambridge University Press, NY, 2003.

D. Avanessoff, A. R. Fioravanti, C. Bonnet. YALTA: a Matlab toolbox for the H_∞-stability analysis of classical and fractional systems with commensurate delays. *IFAC Proceedings Volumes*, vol. 46, No. 2, pp. 839-844, (5th IFAC Symposium on System, Structure and Control, Grenoble, France), February 2013.

D. Avanessoff, C. Bonnet, H. Cavalera, A. R. Fioravanti, L. H. V. Nguyen. *User document YALTA, v.1.0.1.* `https://gforge.inria.fr/projects/yalta-toolbox`, September 2015.

J. A. Burns, T. L. Herdman, and L. Zietsman. Control of PDE systems with delays. *IFAC Proceedings Volumes*, Vol. 46, No. 26, pp. 79-84, (1st IFAC Workshop on Control of Systems Governed by Partial Differential Equations, Paris, France), September 2013.

C. I. Byrnes, M. W. Spong, and T-J. Tarn A several complex variables approach to feedback stabilization of linear neutral delay-differential systems. *Mathematical Systems Theory*, 17:97–133, 1984.

T. T. Georgiou, and M. C. Smith. Robust control of feedback systems with combined plant and controller uncertainty. *Proc. 1990 American Control Conference*, San Diego, CA, pp. 2009–2013, May 1990.

S. Gumussoy. Coprime-inner/outer factorization of SISO time-delay systems and FIR structure of their optimal H-infinity controllers. *Int. J. Robust. Nonlinear Control* 22:981–998, 2012.

S. Gumussoy, and H. Özbay. On the mixed sensitivity minimization for systems with infinitely many unstable modes. *Systems & Control Letters*, 53:211–216, 2004.

J. J. Loiseau, M. Cardelli, and X. Dusser. Neutral time-delay systems that are not formally stable are not BIBO stabilizable. *IMA J. of Mathematical Control and Information*, 19:217–227, 2002.

W. Michiels, and S.-I. Niculescu. *Stability and stabilization of time-delay systems. An eigenvalue based approach.* SIAM, Philadelphia PA, 2007.

L. H. V. Nguyen, and C. Bonnet. Stabilization of some fractional neutral delay systems which possibly possess an infinite number of unstable poles. In: A. Seuret *et al.* (eds) *Low-Complexity Controllers for Time-Delay Systems*, Advances in Delays and Dynamics, vol 2., pp. 47–60, Springer, Cham, 2014.

L. H. V. Nguyen, and C. Bonnet. H_∞-stability analysis of various classes of neutral systems with commensurate delays and with chains of poles approaching the imaginary axis. *Proc. of the 54th IEEE Conference on Decision and Control (CDC2015)*, Osaka, Japan, pp. 6416–6421, December 2015.

L. H. V. Nguyen, C. Bonnet, A. R. Fioravanti. H_∞-stability analysis of fractional delay systems of neutral type, *SIAM J. Control Optim.*, 54:740–759, 2016.

H. Özbay, M. C. Smith, and A. Tannenbaum. Controller design for unstable distributed systems. *Proc. 1990*

American Control Conference, San Diego CA, pp. 1583–1588, May 1990.

J. R. Partington, and C. Bonnet. H_∞ and BIBO stabilization of delay systems of neutral type. *Systems & Control Letters*, 53:283–288, 2004.

R. Rabah, G. M. Sklyar, and A. V. Rezounenko. Stability analysis of neutral type systems in Hilbert space. *J. Differential Equations* 214:391–428, 2005.

R. Rabah, G. M. Sklyar, and A. V. Rezounenko. On strong regular stabilizability for linear neutral type systems. *J. Differential Equations* 245:569–593, 2008.

M. C. Smith. On stabilization and existence of coprime factorizations. *IEEE Trans. on Automatic Control*, 34:1005–1007, 1989.

T. Vyhlídal, and P. Zítek. Mapping based algorithm for large-scale computation of quasi-polynomial zeros. *IEEE Transactions on Automatic Control*, 54:171–177, 2009.

T. Vyhlídal, and P. Zítek. QPmR - Quasi-Polynomial Root-Finder: Algorithm Update and Examples. In T. Vyhlidal, J.F. Lafay, R. Sipahi (eds) *Delay Systems.* Advances in Delays and Dynamics, vol 1, pp. 299–312, Springer, Cham, 2014.

Y. Yamamoto. Pseudo-rational input/output maps and their realizations: a fractional representation approach to infinite-dimensional systems. *SIAM J. Control and Optimization*, 26:1415–1430, 1988.

Y. Yamamoto, and S. Hara. Relationships between internal and external stability for infinite-dimensional systems with applications to a servo problem. *IEEE Transactions on Automatic Control*, vol. 33 (1988), pp. 1044–1052.