Finite-Time Control for Switched Linear Systems with Interval Time-Delay

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Abstract: In this study, finite-time control of switched linear systems with interval time-delay is considered. State feedback is applied in order to ensure finite-time boundedness of the system. Sufficient conditions and average dwell-time bounds are obtained. Because of non-convex terms in the average dwell-time constraint, a technique which converts the nonlinear terms into linear matrix inequality conditions is expressed in terms of the cone-complementarity linearization method. Finally, numerical examples are given for the effectiveness and validity of the proposed solutions.

Keywords: Switched systems, time-delay systems, finite-time boundedness, average dwell-time, cone-complementarity linearization,

1. INTRODUCTION

Switched systems are class of hybrid systems which have a switching sequence directing the system among the finite number of subsystems. They are used for modelling various control problems such as network control, traffic control, process control etc.

Stability is one of the basic research topic for switched systems, which has attracted most of the attention in last decades, Liberzon and Morse (1999); Sun and Ge (2005); Sun (2006); Lin and Antsaklis (2009). Most of the studies related to stability of switched systems focus on Lyapunov asymptotic stability, which is defined over an infinite time interval. However, in many practical applications, finite-time (FT) stability of a system is the main concern, which means keeping the system behavior/state within specified bounds in a fixed FT interval, Dorato (1961); Michel and Wu (1969); Weiss and Infante (1967). FT stability for switched systems is an emerging concept in recent years, Du et al. (2009); Xiang and Xiao (2011, 2013).

Average-dwell time (ADT) is another research topic for switched systems. ADT means that the number of switching instants in a finite interval is bounded and the average time between consecutive switching instants is not less than a constant. The analysis of the switched systems with dwell time Cheng et al. (2015); Karabacak et al. (2014); Zhang and Shi (2009) became prominent after Hespanha and Morse (1999). In the most of the existing literature, a suitable Lyapunov functional is determined to obtain an ADT bound for the stability and the stabilization of switched systems as small as possible.

Time-delay systems have drawn attention to many scholars. If both upper and lower bounds on time-delay exist, such systems are called interval time-delay systems, Botmart et al. (2011); Phat et al. (2012); Shao (2009); Shao and Han (2012). The current methods of stabilization are divided into two categories: delay-dependent Park (1999); Zhang et al. (2005) and delay-independent Zhang et al. (2007). Robust Blizorukova et al. (2001); Busłowicz (2010), H_∞ Fridman and Shaked (2002) and observer based control Kwon et al. (2006) problems are also examined in time-delay systems.

This paper deals with the design of the state-feedback controller to stabilize the interval time-delay switched systems in FT. Some sufficient conditions and new ADT bounds are introduced. Because of nonlinear terms in the ADT constraint, a technique which converts the nonlinear terms into LMI conditions is expressed in terms of the cone-complementarity linearization method.

The notation used in this paper is fairly standard. "*" in a matrix means to be the symmetric term of the corresponding upper triangular element, $\mathcal{C}^1$ is the class of continuously differentiable functions and $\lambda_{max}(A)$ (respectively $\lambda_{min}(A)$) represents the maximum (minimum) eigenvalue of A. Matrices, if not stated, are assumed to have compatible dimensions for algebraic operations.

2. PROBLEM STATEMENT

Consider a switched linear system with an interval time-varying delay in the state vector, where

$$\begin{aligned}\dot{x}(t) =& A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-h(t)) + B_{\sigma(t)}u(t)\\ &+ B_{w\sigma(t)}w(t),\end{aligned} \tag{1}$$

with the initial conditon function

$$x(t) = \phi(t),\ t \in [-h_2, 0]. \tag{2}$$

Here $x(t) \in \mathbf{R}^n$ is the state vector and $u(t) \in \mathbf{R}^m$ the control input, respectively. $A_{\sigma(t)}$, $A_{d\sigma(t)}$, $B_{\sigma(t)}$ and $B_{w\sigma(t)}$ are real constant matrices of appropriate dimensions, $\phi(t) \in \mathcal{C}^1([-h_2, 0], \mathbf{R}^n)$ is the initial function and $h(t) \in \mathcal{C}^1([h_1, h_2], \mathbf{R})$ is the delay satisfying

$$0 \le h_1 \le h(t) \le h_2, \quad \dot{h}(t) \le h_d < \infty. \tag{3}$$

The switching signal is defined as $\sigma(t) : [0,\infty) \to \mathcal{I} = \{1, 2, ..., N\}$ with the switching sequence $\Sigma = \{x_0; (i_0, t_0), (i_1, t_1), ..., (i_k, t_k), ...|i_k \in \mathcal{I}, k = 0, 1, ...\}$, i.e.

$i_k{}^{th}$ system is activated when $t \in [t_k, t_{k+1})$. In this paper, the expression "Switched Systems with Stable Subsystems" means that $A_1, A_2, ..., A_N$ are all Hurwitz stable. $w(t)$ is the exogenous disturbance and satisfies

$$\int_0^\infty w^T(t)w(t)dt < d,\ d \geq 0 \tag{4}$$

Consider the control law

$$u(t) = -K_{\sigma(t)}x(t). \tag{5}$$

The closed-loop system is given as follows

$$\begin{aligned}\dot{x}(t) =& A_{K\sigma(t)}x(t) + A_{d\sigma(t)}x(t-h(t)) \\ &+ B_{w\sigma(t)}w(t),\end{aligned} \tag{6}$$

where $A_{K\sigma(t)} = A_{\sigma(t)} - B_{\sigma(t)}K_{\sigma(t)}$.

Lemma 1. (Schur complement) Given constant matrices S_{11}, S_{12}, S_{22} with appropriate dimensions satisfying $S_{11} = S_{11}^T$ and $S_{22} = S_{22}^T$ and $S_{22} < 0$, the LMI $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22}\end{bmatrix} < 0$ is equivalent to $S_{11} + S_{12}S_{22}^{-1}S_{12}^T < 0$, Boyd and Vandenberghe (2004).

Lemma 2. (Grönwall's lemma) If a differentiable function $\psi(t) > 0$ on the open interval $U = (a, b)$ (as well as $U = [a, b]$ or $U = [a, b)$) and

$$\dot{u}(t) \leq \phi(t) + \psi(t)u(t)$$

then

$$u(t) \leq u(a)e^{\Psi(t)} + \int_a^t \phi(s)e^{\Psi(t)-\Psi(s)}ds$$

for $t < b$ where

$$\Psi(t) = \int_a^t \psi(s)ds,$$

Perko (2013).

Lemma 3. (Jensen's inequality) For any symmetric positive definite matrix $M > 0$, scalars $a, b > 0$ with $b > a$ and an integrable vector function $x : [a, b] \to \mathbf{R}^n$, the following inequality holds, Gu et al. (2003).

$$\begin{aligned}&\left(\int_a^b x(s)ds\right)^T M \left(\int_a^b x(s)ds\right) \\ &\leq (b-a)\left(\int_a^b x^T(s)Mx(s)ds\right)\end{aligned}$$

Definition 1. Let $N_{\sigma(t)}(t, T)$ denotes the switching number of the switching signal $\sigma(t)$ for the interval $0 \leq t \leq T$. N_0 is the chatter bound. Then the following inequality holds

$$N_{\sigma(t)}(t, T) \leq N_0 + (T-t)/\tau_a$$

for so called ADT τ_a, Hespanha and Morse (1999).

Definition 2. (FT Stability) Given scalars $\delta > 0$, $\epsilon > 0$, $T_f > 0$ with $0 \leq \delta \leq \epsilon$ and a matrix $R > 0$ with appropriate dimensions, the switched system (1) with $u(t) \equiv 0$ and $w(t) \equiv 0$ is said to be FT stable with respect to $(\delta, \epsilon, T_f, R)$, if $\sup_{s\in[-h_2,0]}\left\{x^T(s)Rx(s)\right\} < \delta$ then $x^T(t)Rx(t) < \epsilon$, $\forall t \in [0, T_f]$, Liu et al. (2012).

Definition 3. (FT Boundedness) Given scalars $\delta > 0$, $\epsilon > 0$, $T_f > 0$ with $0 \leq \delta \leq \epsilon$ and a matrix $R > 0$ with appropriate dimensions, the switched system (1) with $u(t) \equiv 0$ is said to be FT bounded $(\delta, \epsilon, T_f, d, R)$, if $\sup_{s\in[-h_2,0]}\left\{x^T(s)Rx(s)\right\} < \delta$ then $x^T(t)Rx(t) < \epsilon$, $\forall t \in [0, T_f]$ and $\forall w(t) : \int_0^{T_f} w^T(t)w(t)dt < d$, Liu et al. (2012).

3. FT BOUNDEDNESS ANALYSIS

In this section, we suppose that A_1, A_2,..., A_r, $(1 \leq r < N)$ in system (1) are Hurwitz stable and the remaining matrices are unstable. Let us define

$$\psi_i = \begin{cases} -\alpha_i & i \in \mathcal{I}_{st} \\ \alpha_i & i \in \mathcal{I}_{un}\end{cases}$$

where $\mathcal{I}_{st}$ and $\mathcal{I}_{un}$ are the index set of all Hurwitz stable and unstable subsystems, respectively. Note that $\mathcal{I} = \mathcal{I}_{st} \cup \mathcal{I}_{un}$. For a given switching sequence Σ, the total activation times of stable and unstable subsystems are defined as T^- and T^+, respectively in a finite interval $[0, T_f]$. Thus, $T_f = T^+ + T^-$.

Theorem 4. Consider the switched system (1) with r Hurwitz stable and $N - r$ unstable subsystems. The system (1) is FT bounded with respect to $(\delta, \epsilon, T_f, d, R)$, for given constants $\alpha_i \geq 0$, $\mu \geq 1$, $T^+ > 0$ and $T^- > 0$ such that $T_f = T^+ + T^-$, if there exist a set of symmetric matrices for every i^{th} system $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $S_{1i} > 0$, $S_{2i} > 0$, $T_i > 0$, $W_i > 0$, Y_i, M_{1i}, M_{2i}, N_{1i}, N_{2i} satisfying

$$\Upsilon_i = \begin{bmatrix} \Omega_i & -M_i & -N_i & Z_i \\ * & -e^{2\psi_i h_2}S_{2i} & 0 & 0 \\ * & * & -e^{2\psi_i h_2}S_{2i} & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{7a}$$

$$e^{2\alpha_{max}^+ T^+}\eta'_+ \leq \lambda_1 e^{2\alpha_{min}^- T^-}\epsilon \tag{7b}$$

$$P_j \leq \mu P_i,\ Q_{kj} \leq \mu Q_{ki}, S_{kj} \leq \mu S_{ki}, T_j \leq \mu T_i, \tag{7c}$$

for $i, j \in \mathcal{I}$ and $k = 1, 2$, where

$$\Omega_i = \begin{bmatrix} \Omega_{11,i} & \Omega_{12,i} & \Omega_{13,i} & -N_{1i} & B_{wi} & \Omega_{16,i} \\ * & \Omega_{22,i} & M_{2i} & -N_{2i} & 0 & P_iA_{di}^T \\ * & * & \Omega_{33,i} & 0 & 0 & 0 \\ * & * & * & \Omega_{44,i} & 0 & 0 \\ * & * & * & * & -W_i & B_{wi}^T \\ * & * & * & * & * & \Omega_{66,i} \end{bmatrix} \tag{8}$$

with entries

$$\begin{aligned}\Omega_{11,i} =& A_iP_i + P_iA_i^T - B_iY_i - Y_i^TB_i^T + Q_{1i} + Q_{2i} \\ &- e^{2\psi_i h_1}S_{1i} - 2\psi_iP_i + T_i, \\ \Omega_{12,i} =& A_{di}P_i - M_{1i} + N_{1i}, \\ \Omega_{13,i} =& e^{2\psi_i h_1}S_{1i} + M_{1i}, \\ \Omega_{16,i} =& P_iA_i^T - Y_i^TB_i^T, \\ \Omega_{22,i} =& N_{2i} + N_{2i}^T - M_{2i} - M_{2i}^T - (1-h_d)e^{2\psi_i h_2}T_i \\ \Omega_{33,i} =& -e^{2\psi_i h_1}(Q_{1i} + S_{1i}), \\ \Omega_{44,i} =& -e^{2\psi_i h_2}Q_{2i}, \\ \Omega_{66,i} =& h_1^2S_{1i} + h_{12}^2S_{2i} - 2P_i.\end{aligned}$$

Then the ADT of the switching signal satisfies

$$\tau_a > \\ \tau_a^* = \frac{T_f \ln\mu}{\ln(\lambda_1\epsilon) - \ln\eta'_+ - 2\alpha_{max}^+T^+ + 2\alpha_{min}^-T^- - N_0\ln\mu} \tag{9}$$

where $\alpha_{max}^+ = \max_{i\in\mathcal{I}_{un}}\{\alpha_i\}$, $\alpha_{min}^- = \min_{i\in\mathcal{I}_{st}}\{\alpha_i\}$ and

$$\begin{aligned}\eta'_+ =& \lambda_2\delta + \lambda'_3h_1e^{2\alpha_{max}^+h_1}\delta + \lambda'_4h_2e^{2\alpha_{max}^+h_2}\delta \\ &+ \lambda'_5h_1^3e^{2\alpha_{max}^+h_1}\delta' \\ &+ \lambda'_6h_{12}^2(h_1e^{2\alpha_{max}^+h_1} + h_{12}e^{2\alpha_{max}^+h_2})\delta' \\ &+ \lambda'_7h_2e^{2\alpha_{max}^+h_2}\delta + \lambda_8 d.\end{aligned} \tag{10}$$

with matrix transformations

$$
\begin{aligned}
\hat{Q}_{1i} =& R^{1/2}Q_{1i}R^{1/2},\ \hat{Q}_{2i} = R^{1/2}Q_{2i}R^{1/2},\\
\hat{S}_{1i} =& R^{1/2}S_{1i}R^{1/2},\ \hat{S}_{2i} = R^{1/2}S_{2i}R^{1/2},\\
\hat{T}_i =& R^{1/2}T_iR^{1/2},\\
Q_{1i} =& P_i\overline{Q}_{1i}P_i,\ Q_{2i} = P_i\overline{Q}_{2i}P_i,\\
S_{1i} =& P_i\overline{S}_{1i}P_i,\ S_{2i} = P_i\overline{S}_{2i}P_i,\\
T_i =& P_i\overline{T}_iP_i,\\
M_{1i} =& P_i\overline{M}_{1i}P_i,\ M_{2i} = P_i\overline{M}_{2i}P_i,\\
N_{1i} =& P_i\overline{N}_{1i}P_i,\ N_{2i} = P_i\overline{N}_{2i}P_i
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
\lambda_1 =& \inf_{i\in\mathcal{I}}\{\lambda_{min}(\tilde{P}_i^{-1})\},\ \lambda_2 = \sup_{i\in\mathcal{I}}\{\lambda_{max}(\tilde{P}_i^{-1})\},\\
\lambda_3' =& \sup_{i\in\mathcal{I}}\{\lambda_{max}(\tilde{P}_i^{-1}\hat{Q}_{1i}\tilde{P}_i^{-1})\},\\
\lambda_4' =& \sup_{i\in\mathcal{I}}\{\lambda_{max}(\tilde{P}_i^{-1}\hat{Q}_{2i}\tilde{P}_i^{-1})\},\\
\lambda_5' =& \sup_{i\in\mathcal{I}}\{\lambda_{max}(\tilde{P}_i^{-1}\hat{S}_{1i}\tilde{P}_i^{-1})\},\\
\lambda_6' =& \sup_{i\in\mathcal{I}}\{\lambda_{max}(\tilde{P}_i^{-1}\hat{S}_{2i}\tilde{P}_i^{-1})\},\\
\lambda_7' =& \sup_{i\in\mathcal{I}}\{\lambda_{max}(\tilde{P}_i^{-1}\hat{T}_i\tilde{P}_i^{-1})\},\\
\lambda_8 =& \sup_{i\in\mathcal{I}}\{\lambda_{max}(W_i)\},\ \delta' = \sup_{s\in[-h_2,0]}\{\dot{x}^T(s)R\dot{x}(s)\}\\
h_{12} =& h_2 - h_1,\ Z_i = [0\ 0\ 0\ 0\ C_iP_i\ 0\ 0]^T\\
M_i =& [M_{1i}\ M_{2i}\ 0\ 0\ 0\ 0\ 0]^T,\ N_i = [N_{1i}\ N_{2i}\ 0\ 0\ 0\ 0\ 0]^T.
\end{aligned}
$$

The gain matrices K_i and L_i of controller and observer are perceived as

$$
K_i = Y_iP_i^{-1},\ L_i = -\frac{1}{2}P_iC_i^T \tag{12}
$$

Proof. Consider the following Lyapunov-Krasovskii candidate functional as

$$
V_i(x(t)) = \sum_{j=1}^{6} V_{ji}(x(t)) \tag{13}
$$

where

$$
\begin{aligned}
V_{1i}(x(t)) =& x^T(t)P_i^{-1}x(t)\\
V_{2i}(x(t)) =& \int_{t-h_1}^{t} e^{2\psi_i(t-s)}x^T(s)\overline{Q}_{1i}x(s)ds\\
V_{3i}(x(t)) =& \int_{t-h_2}^{t} e^{2\psi_i(t-s)}x^T(s)\overline{Q}_{2i}x(s)ds\\
V_{4i}(x(t)) =& \int_{-h_1}^{0}\int_{t+\theta}^{t} h_1e^{2\psi_i(t-s)}\dot{x}^T(s)\overline{S}_{1i}\dot{x}(s)dsd\theta\\
V_{5i}(x(t)) =& \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} h_{12}e^{2\psi_i(t-s)}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)dsd\theta\\
V_{6i}(x(t)) =& \int_{t-h(t)}^{t} e^{2\psi_i(t-s)}x^T(s)\overline{T}_ix(s)ds
\end{aligned}
\tag{14}
$$

The derivatives are obtained as follows

$$
\begin{aligned}
\dot{V}_{1i}(x(t)) =& x^T(t)[P_i^{-1}A_{Ki} + A_{Ki}^TP_i^{-1}]x(t)\\
&+ 2x^T(t)P_i^{-1}A_{di}x(t-h(t))\\
&+ 2x^T(t)P_i^{-1}B_{wi}w(t)\\
\dot{V}_{2i}(x(t)) =& 2\psi_iV_{2i} + x^T(t)\overline{Q}_{1i}x(t)\\
&- e^{2\psi_ih_1}x^T(t-h_1)\overline{Q}_{1i}x(t-h_1)\\
\dot{V}_{3i}(x(t)) =& 2\psi_iV_{3i} + x^T(t)\overline{Q}_{2i}x(t)\\
&- e^{2\psi_ih_2}x^T(t-h_2)\overline{Q}_{2i}x(t-h_2)\\
\dot{V}_{4i}(x(t)) =& 2\psi_iV_{4i} + h_1^2\dot{x}^T(t)\overline{S}_{1i}\dot{x}(t)\\
&- e^{2\psi_ih_1}\int_{t-h_1}^{t} h_1\dot{x}^T(s)\overline{S}_{1i}\dot{x}(s)ds\\
\dot{V}_{5i}(x(t)) \le& 2\psi_iV_{5i} + h_{12}^2\dot{x}^T(t)\overline{S}_{2i}\dot{x}(t)\\
&- e^{2\psi_ih_2}\int_{t-h_2}^{t-h_1} h_{12}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds\\
\dot{V}_{6i}(x(t)) \le& 2\psi_iV_{6i} + x^T(t)\overline{T}_ix(t)\\
&- (1-h_d)e^{2\psi_ih_2}x^T(t-h(t))\overline{T}_ix(t-h(t))
\end{aligned}
\tag{15}
$$

By Jensen's Inequality, $\dot{V}_{4i}(x(t))$ can be written as

$$
\begin{aligned}
\dot{V}_{4i}(x(t)) \le& 2\psi_iV_{4i}(x(t)) + h_1^2\dot{x}^T(t)\overline{S}_{1i}\dot{x}(t)\\
&- e^{2\psi_ih_1}x^T(t)\overline{S}_{1i}x(t)\\
&+ 2e^{2\psi_ih_1}x^T(t)\overline{S}_{1i}x(t-h_1)\\
&- e^{2\psi_ih_1}x^T(t-h_1)\overline{S}_{1i}x(t-h_1)
\end{aligned}
\tag{16}
$$

From (3), it is clear that $-(h_2-h_1) \le -(h_2-h(t))$ and $-(h_2-h_1) \le -(h(t)-h_1)$. Thus

$$
\begin{aligned}
&-h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds \le\\
&- (h_2-h(t))\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds\\
&- (h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds
\end{aligned}
\tag{17}
$$

Let $\int_{t-h(t)}^{t-h_1}\dot{x}(s)ds =: i_{h_1}(t)$ and $\int_{t-h_2}^{t-h(t)}\dot{x}(s)ds =: i_{h_2}(t)$. Then, by Jensen's Inequality, (17) is written as follows

$$
\begin{aligned}
&-h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds\\
&\le -i_{h_2}^T(t)\overline{S}_{2i}i_{h_2}(t) - i_{h_1}^T(t)\overline{S}_{2i}i_{h_1}(t)
\end{aligned}
\tag{18}
$$

Now, define

$$
\begin{aligned}
\xi(t) = \big[&x^T(t)\ x^T(t-h(t))\ x^T(t-h_1)\ x^T(t-h_2)\\
&w^T(t)\ \dot{x}^T(t)\ i_{h_1}^T(t)\ i_{h_2}^T(t)\big]^T.
\end{aligned}
\tag{19}
$$

By Leibniz's formula, we have

$$
\begin{aligned}
2\xi^T(t)\mathcal{M}_i\Big[x(t-h_1) - x(t-h(t)) - i_{h_1}(t)\Big] = 0\\
2\xi^T(t)\mathcal{N}_i\Big[x(t-h(t)) - x(t-h_2) - i_{h_2}(t)\Big] = 0
\end{aligned}
\tag{20}
$$

Also from (6), it can be written

$$
\begin{aligned}
2\dot{x}^T(t)P_i^{-1}\Big[&A_{Ki}x(t) + A_{di}x(t-h(t))\\
&+ B_{wi}w(t) - \dot{x}(t)\Big] = 0
\end{aligned}
\tag{21}
$$

On the other hand, for a positive definite matrix W_i the following holds

$$\left[w^T(t)W_iw(t) - w^T(t)W_iw(t)\right] = 0 \tag{22}$$

Then, by the equations (13)-(21), we obtain

$$\dot{V}_i(x(t)) - 2\psi_i V_i(x(t)) \leq \xi^T(t)\Sigma_i\xi(t) + w^T(t)W_iw(t). \tag{23}$$

Here

$$\Sigma_i = \begin{bmatrix} \Xi_i & -\mathcal{M}_i & -\mathcal{N}_i \\ * & -e^{2\psi_i h_2}\overline{S}_{2i} & 0 \\ * & * & -e^{2\psi_i h_2}\overline{S}_{2i} \end{bmatrix} \tag{24}$$

for $i, j \in \mathcal{I}$ and $k = 1, 2$, where

$$\Xi_i = \begin{bmatrix} \Xi_{11,i} & \Xi_{12,i} & \Xi_{13,i} & -\overline{N}_{1i} & P_i^{-1}B_{wi} & A_{Ki}^TP_i^{-1} \\ * & \Xi_{22,i} & \overline{M}_{2i} & -\overline{N}_{2i} & 0 & A_{di}^TP_i^{-1} \\ * & * & \Xi_{33,i} & 0 & 0 & 0 \\ * & * & * & \Xi_{44,i} & 0 & 0 \\ * & * & * & * & -W_i & B_{wi}^TP_i^{-1} \\ * & * & * & * & * & \Xi_{66,i} \end{bmatrix} \tag{25}$$

with entries

$$\begin{aligned}
\Xi_{11,i} =& P_i^{-1}A_{Ki} + A_{Ki}^TP_i^{-1} + \overline{Q}_{1i} + \overline{Q}_{2i} - e^{2\psi_i h_1}\overline{S}_{1i} \\
& - 2\psi_i P_i^{-1} + \overline{T}_i, \\
\Xi_{12,i} =& P_i^{-1}A_{di} - \overline{M}_{1i} + \overline{N}_{1i}, \\
\Xi_{13,i} =& e^{2\psi_i h_1}\overline{S}_{1i} + \overline{M}_{1i}, \\
\Xi_{22,i} =& \overline{N}_{2i} + \overline{N}_{2i}^T - \overline{M}_{2i} - \overline{M}_{2i}^T - (1-h_d)e^{2\psi_i h_2}\overline{T}_i, \\
\Xi_{33,i} =& -e^{2\psi_i h_1}(\overline{Q}_{1i} + \overline{S}_{1i}), \\
\Xi_{44,i} =& -e^{2\psi_i h_2}\overline{Q}_{2i}, \\
\Xi_{66,i} =& h_1^2\overline{S}_{1i} + h_{12}^2\overline{S}_{2i} - 2P_i^{-1} \\
\mathcal{M}_i =& [\overline{M}_{1i}^T \ \overline{M}_{2i}^T \ 0\ 0\ 0\ 0\ 0]^T, \ \mathcal{N}_i = [\overline{N}_{1i}^T \ \overline{N}_{2i}^T \ 0\ 0\ 0\ 0\ 0]^T
\end{aligned}$$

By pre- and post-multiplying both sides of the Inequalities in (24) with (25) by $\mathcal{D}_i = diag\{P_i, P_i, P_i, P_i, I, P_i, P_i, P_i\}$, Υ_i of (7a) are obtained. From (7a)

$$\dot{V}_i(x(t)) - 2\psi_i V_i(x(t)) \leq w^T(t)W_iw(t) \tag{26}$$

is obtained.

On the other hand, by applying Grönwall's Lemma on $t \in [t_k, t_{k+1})$ we have

$$\begin{aligned}
V_{\sigma(t)}(x(t)) \leq & e^{2\psi_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}(x(t_k)) \\
& + \int_{t_k}^{t} e^{2\psi_{\sigma(t_k)}(t-s)}w^T(s)W_{\sigma(t_k)}w(s)ds.
\end{aligned} \tag{27}$$

Consider (7c) and assume $\sigma(t_k) = i$ and $\sigma(t_k^-) = j$, we have

$$V_{\sigma(t_k)}(x(t_k)) \leq \mu V_{\sigma(t_k^-)}(x(t_k^-)) \tag{28}$$

If Grönwall's Lemma and (28) is applied to (26) until $[0, t_1)$ iteratively, we get

$$\begin{aligned}
V_{\sigma(t)}(x(t)) \leq & e^{2\psi_{\sigma(t_k)}(t-t_k)+\ldots+2\psi_{\sigma(0)}(t_1-0)} \\
& \times \mu^N V_{\sigma(0)}(x(0)) \\
& + \mu^N \int_0^{t_1} e^{2\psi_{\sigma(t_k)}(t-t_k)+\ldots+2\psi_{\sigma(0)}(t_1-s)} \\
& \times w^T(s)W_{\sigma(0)}w(s)ds \\
& + \ldots \\
& + \int_{t_k}^{t} e^{2\psi_{\sigma(t_k)}(t-s)}w^T(s)W_{\sigma(t_k)}w(s)ds
\end{aligned} \tag{29}$$

By considering the activation times T^- and T^+ for stable and unstable subsystems, respectively, the inequality (29) can be written as follows:

$$\begin{aligned}
V_{\sigma(t)}(x(t)) \leq & e^{2\alpha_{max}^+T^+ - 2\alpha_{min}^-T^-} \\
& \times \mu^N\left(V_{\sigma(0)}(x(0)) + \lambda_8 d\right).
\end{aligned} \tag{30}$$

where N denotes the switching number of $\sigma(t)$ over $(0, T_f)$. Moreover,

$$\begin{aligned}
V_{\sigma(t)}(x(0)) =& x^T(0)P_{\sigma(0)}^{-1}x(0) \\
& + \int_{-h_1}^{0} e^{-2\psi_{\sigma(0)}s}x^T(s)\overline{Q}_{1\sigma(0)}x(s)ds \\
& + \int_{-h_2}^{0} e^{-2\psi_{\sigma(0)}s}x^T(s)\overline{Q}_{2\sigma(0)}x(s)ds \\
& + \int_{-h_1}^{0}\int_{\theta}^{0} h_1 e^{-2\psi_{\sigma(0)}s}\dot{x}^T(s)\overline{S}_{1\sigma(0)}\dot{x}(s)dsd\theta \\
& + \int_{-h_2}^{-h_1}\int_{\theta}^{0} h_{12} e^{-2\psi_{\sigma(0)}s}\dot{x}^T(s)\overline{S}_{2\sigma(0)}\dot{x}(s)dsd\theta \\
& + \int_{-h(0)}^{0} e^{-2\psi_{\sigma(0)}s}x^T(s)\overline{T}_{\sigma(0)}x(s)ds.
\end{aligned} \tag{31}$$

When the orders of the double integrals are changed and the matrices in (11) are substituted, we have

$$\begin{aligned}
V_{\sigma(0)}(x(0)) =& x^T(0)P_{\sigma(0)}^{-1}x(0) \\
& + \int_{-h_1}^{0} e^{-2\psi_{\sigma(0)}s}x^T(s)P_{\sigma(0)}^{-1}Q_{1\sigma(0)}P_{\sigma(0)}^{-1}x(s)ds \\
& + \int_{-h_2}^{0} e^{-2\psi_{\sigma(0)}s}x^T(s)P_{\sigma(0)}^{-1}Q_{2\sigma(0)}P_{\sigma(0)}^{-1}x(s)ds \\
& + \int_{-h_1}^{0}\int_{-h_1}^{s} h_1 e^{-2\psi_{\sigma(0)}s} \\
& \qquad \times \dot{x}^T(s)P_{\sigma(0)}^{-1}S_{1\sigma(0)}P_{\sigma(0)}^{-1}\dot{x}(s)d\theta ds \\
& + \int_{-h_2}^{-h_1}\int_{-h_2}^{s} h_{12} e^{-2\psi_{\sigma(0)}s} \\
& \qquad \times \dot{x}^T(s)P_{\sigma(0)}^{-1}S_{2\sigma(0)}P_{\sigma(0)}^{-1}\dot{x}(s)d\theta ds \\
& + \int_{-h_1}^{0}\int_{-h_2}^{-h_1} h_{12} e^{-2\psi_{\sigma(0)}s} \\
& \qquad \times \dot{x}^T(s)P_{\sigma(0)}^{-1}S_{2\sigma(0)}P_{\sigma(0)}^{-1}\dot{x}(s)d\theta ds \\
& + \int_{-h(0)}^{0} e^{-2\psi_{\sigma(0)}s}x^T(s)P_{\sigma(0)}^{-1}T_{\sigma(0)}P_{\sigma(0)}^{-1}x(s)ds.
\end{aligned}$$

From (11), each matrix can be bounded as

$$\begin{aligned}
P_{\sigma(0)}^{-1}Q_{1\sigma(0)}P_{\sigma(0)}^{-1} =& R^{1/2}\tilde{P}_{\sigma(0)}^{-1}\hat{Q}_{1\sigma(0)}\tilde{P}_{\sigma(0)}^{-1}R^{1/2} \\
\leq & \lambda_{max}\left(\tilde{P}_{\sigma(0)}^{-1}\hat{Q}_{1\sigma(0)}\tilde{P}_{\sigma(0)}^{-1}\right)R \leq \lambda_3' R.
\end{aligned}$$

Also, note that

$$\begin{aligned}
\sup_{s\in[-h(0),0]}\{e^{-2\psi_{\sigma(0)}s}\} \leq \sup_{s\in[-h_1,0]}\{e^{-2\psi_{\sigma(0)}s}\} &= e^{2\alpha_{max}^+h_1}, \\
\sup_{s\in[-h_2,0]}\{e^{-2\psi_{\sigma(0)}s}\} &= e^{2\alpha_{max}^+h_2}
\end{aligned} \tag{32}$$

Here, an upper bound for $V_{\sigma(0)}(0)$ can be written as follows

$$V_{\sigma(0)}(x(0)) \leq \lambda_2\delta + \lambda'_3 h_1 e^{2\alpha^+_{max}h_1}\delta + \lambda'_4 h_2 e^{2\alpha^+_{max}h_2}\delta + \lambda'_5 h_1^3 e^{2\alpha^+_{max}h_1}\delta' + \lambda'_6 h_{12}^2 (h_1 e^{2\alpha^+_{max}h_1} + h_{12}e^{2\alpha^+_{max}h_2})\delta' + \lambda'_7 h_2 e^{2\alpha^+_{max}h_2}\delta. \tag{33}$$

Since,

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\geq x^T(t)P_i^{-1}x(t) = x^T(t)R^{1/2}\tilde{P}_i^{-1}R^{1/2}x(t) \\ &\geq \inf_{i\in I}\left(\lambda_{min}(\tilde{P}_i^{-1})\right)x^T(t)Rx(t) \\ &= \lambda_1 x^T(t)Rx(t). \end{aligned} \tag{34}$$

By the equations (30), (33) and (34) the inequality $x^T(t)Rx(t) < \epsilon$ is obtained, which tells that the switched system (1) is FT bounded. Then, for $\mu = 1$ the inequality in (7b) and for $\mu > 1$ the ADT bound in (9) are calculated. □

Remark 1. Note that the condition (7b) contains the constants λ_1, λ_2, λ'_3, λ'_4, λ'_5,λ'_6, λ'_7 and λ_8. The existance of these constants depends on the solutions of the following inequalities

$$\begin{aligned} &\lambda_1 I < \tilde{P}_i^{-1} < \lambda_2 I \\ &0 < \tilde{P}_i^{-1}\hat{Q}_{1i}\tilde{P}_i^{-1} < \lambda'_3 I,\ 0 < \tilde{P}_i^{-1}\hat{Q}_{2i}\tilde{P}_i^{-1} < \lambda'_4 I \\ &0 < \tilde{P}_i^{-1}\hat{S}_{1i}\tilde{P}_i^{-1} < \lambda'_5 I,\ 0 < \tilde{P}_i^{-1}\hat{S}_{2i}\tilde{P}_i^{-1} < \lambda'_6 I, \\ &0 < \tilde{P}_i^{-1}\hat{T}_i\tilde{P}_i^{-1} < \lambda'_7 I,\quad 0 < W_i < \lambda_8 I. \end{aligned} \tag{35}$$

For more details see Lin et al. (2011).

To solve the inequalities in (35), it is necessary to put them into LMIs form. Thus, consider $0 < \tilde{P}_i^{-1}\hat{Q}_{1i}\tilde{P}_i^{-1} < \lambda'_3 I$, write it as $-\lambda'_3 I + \tilde{P}_i^{-1}\hat{Q}_{1i}\tilde{P}_i^{-1} < 0$ and use Schur Complement

$$\begin{bmatrix} -\lambda'_3 I & \tilde{P}_i^{-1} \\ * & -\hat{Q}_{1i}^{-1} \end{bmatrix} \leq 0 \iff \begin{bmatrix} -\lambda'_3 I & J_i \\ * & -E_{1i} \end{bmatrix} \leq 0 \tag{36}$$

where $J_i := \tilde{P}_i^{-1}$ and $E_{1i} := \hat{Q}_{1i}^{-1}$ (or equivalently $J_i\tilde{P}_i = I$ and $E_{1i}\hat{Q}_{1i} = I$). By applying same procedure to the other nonlinear inequalities from (35) and defining the matrices E_{2i}, F_{1i}, F_{2i} and G_i for the matrix inverse approximates of $\hat{Q}_{2i}$, $\hat{S}_{1i}$, $\hat{S}_{2i}$ and $\hat{T}_i$, the following inequalities can be stated in terms of cone-complementarity algorithm given in El Ghaoui et al. (1997).

$$\begin{aligned} &\lambda_1 I < J_i < \lambda_2 I,\ 0 \leq \begin{bmatrix} \tilde{P}_i & I \\ * & J_i \end{bmatrix}, \\ &\begin{bmatrix} -\lambda'_3 I & J_i \\ * & -E_{1i} \end{bmatrix} \leq 0, 0 \leq \begin{bmatrix} \hat{Q}_{1i} & I \\ * & E_{1i} \end{bmatrix}, \\ &\begin{bmatrix} -\lambda'_4 I & J_i \\ * & -E_{2i} \end{bmatrix} \leq 0, 0 \leq \begin{bmatrix} \hat{Q}_{2i} & I \\ * & E_{2i} \end{bmatrix}, \\ &\begin{bmatrix} -\lambda'_5 I & J_i \\ * & -F_{1i} \end{bmatrix} \leq 0, 0 \leq \begin{bmatrix} \hat{S}_{1i} & I \\ * & F_{1i} \end{bmatrix}, \\ &\begin{bmatrix} -\lambda'_6 I & J_i \\ * & -F_{2i} \end{bmatrix} \leq 0, 0 \leq \begin{bmatrix} \hat{S}_{2i} & I \\ * & F_{2i} \end{bmatrix}, \\ &\begin{bmatrix} -\lambda'_7 I & J_i \\ * & -G_i \end{bmatrix} \leq 0, 0 \leq \begin{bmatrix} \hat{T}_i & I \\ * & G_i \end{bmatrix}, \\ &0 < W_i < \lambda_8 I, \end{aligned} \tag{37}$$

Algorithm 1. This algorithm is derived for Theorem 4.

- **Step 1:** Find a feasible set

$$(P_i^0, Q_{1i}^0, Q_{2i}^0, S_{1i}^0, S_{2i}^0, T_i^0, J_i^0, E_{1i}^0, E_{2i}^0, F_{1i}^0, F_{2i}^0, G_i^0, W_i^0, T_i^0, M_{1i}^0, M_{2i}^0, N_{1i}^0, N_{2i}^0)$$

  satisfying the inequalities in (7a), (7b), (7c) and (37). Set $k = 0$.
- **Step 2:** Solve the following LMI problem for the variables

$$(P_i, Q_{1i}, Q_{2i}, S_{1i}, S_{2i}, T_i, J_i, E_{1i}, E_{2i}, F_{1i}, F_{2i}, G_i, W_i, T_i, M_{1i}, M_{2i}, N_{1i}, N_{2i})$$

  according to the following minimization problem

$$\begin{aligned} \textit{minimize}\ tr\Bigg(\sum_{i\in\mathcal{I}} &J_i^k\tilde{P}_i + J_i\tilde{P}_i^k + E_{1i}^k\hat{Q}_{1i} + E_{1i}\hat{Q}_{1i}^k \\ &+ E_{2i}^k\hat{Q}_{2i} + E_{2i}\hat{Q}_{2i}^k + F_{1i}^k\hat{S}_{1i} + F_{1i}\hat{S}_{1i}^k \\ &+ F_{2i}^k\hat{S}_{2i} + F_{2i}\hat{S}_{2i}^k + G_i^k\hat{T}_i + G_i\hat{T}_i^k\Bigg) \end{aligned}$$

  subject to (7a), (7b), (7c) *and* (37)
- **Step 3:** If a stopping criteria is satisfied, then exit. Otherwise, set

$$\begin{aligned} &\tilde{P}_i^k = \tilde{P}_i, \hat{Q}_{1i}^k = \hat{Q}_{1i}, \hat{Q}_{2i}^k = \hat{Q}_{2i}, \hat{S}_{1i}^k = \hat{S}_{1i}, \hat{S}_{2i}^k = \hat{S}_{2i}, \\ &\hat{T}_i^k = \hat{T}_i, J_i^k = J_i, E_{1i}^k = E_{1i}, E_{2i}^k = E_{2i}, \\ &F_{1i}^k = F_{1i}, F_{2i}^k = F_{2i}, G_i^k = G_i \end{aligned}$$

  and set $k = k + 1$ and go to Step 3.

4. NUMERICAL EXAMPLES

A numerical example is presented in order to show the effect of the Algorithm 1.

Example 1. Consider the switched system with time delay (1) with two subsystems

$$A_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & -0.34 \end{bmatrix},\ A_2 = \begin{bmatrix} -1.6 & 0 \\ 0 & -0.14 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} -0.06 & 0 \\ 0.06 & -0.03 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -0.03 & 0 \\ -0.69 & -0.12 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.3 \\ 0.15 \end{bmatrix},\ B_{w1} = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix},\ B_{w2} = \begin{bmatrix} 0.15 \\ 0.3 \end{bmatrix}.$$

Note that, A_1 is Hurwitz unstable and A_2 is Hurwitz stable. The activation times of the unstable and unstable subsystems are chosen as $T^+ = 0.6$ and $T^- = 1.4$,, respectively. The constants

$$\psi_1 = 0.5,\ \psi_2 = -0.05,\ h_1 = 0,\ h_2 = 0.1,\ h_d = 0.01,$$
$$R = I,\ \delta = 4,\ \delta' = 4,\ \epsilon = 25,\ \mu = 1.01,\ d = 0.01,$$
$$T_f = 2,\ N_0 = 0.$$

are chosen and by Algorithm 1, we get a feasible solution with controller gains

$$K_1 = [1850.6\ 388.3],\ K_2 = [-662.5\ 1760.7]$$

with the ADT $\tau_a^* = 0.2180$.

5. CONCLUSION

This paper investigates the FT boundedness of the switched systems with interval time-delay and disturbances. Based on a state-feedback controller, some sufficient conditions are obtained for systems vector. Due to the nonconvex elements on these conditions, a cone-complementarity linearization is made. Number of numerical examples exhibit better results with the designed method.

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