Stability of differential-difference systems

Josef Diblík[1]

[1] *Brno University of Technology, Brno, Czech Republic (e-mail: diblik@feec.vutbr.cz)*

The investigation of the stability of linear differential and difference systems with delay is a constant priority of research. We will deal with the exponential stability of linear discrete systems with multiple delays

$$x\,(k+1) = Ax(k) + \sum_{i=1}^{s} B_i x(k-m_i), \quad k = 0, 1, \dots \tag{1}$$

where $s \in \mathbb{N}$, A and B_i are $n \times n$ matrices and $m_i \in \mathbb{N}$. For (1) exponential-type stability as well as exponential estimate of the rate of convergence of solutions are derived. Set $m := \max\{m_1, \dots, m_s\}$. The initial Cauchy problem for the system (1) is

$$x\,(k) = x_k, \;\; k = -m, -m+1, \dots, 0, \;\; x_k \in \mathbb{R}.$$

For a vector $x = (x_1, \dots, x_n)^T$, we define $|x|^2 := \sum\limits_{i=1}^{n} x_i^2$. Let $\rho(A)$ be the spectral radius of the matrix A. Denote by $\lambda_{\max}(\mathcal{A})$ and $\lambda_{\min}(\mathcal{A})$ the maximum, and the minimum eigenvalues respectively of a symmetric matrix $\mathcal{A}$ and define $\varphi(\mathcal{A}) := \lambda_{\max}(\mathcal{A})\lambda_{\min}^{-1}(\mathcal{A})$. For a given matrix $\mathcal{B}$, we use the norm defined by $|\mathcal{B}|^2 := \lambda_{\max}\left(\mathcal{B}^T\mathcal{B}\right)$. Assume $|A| + \sum_{i=1}^{s} |B_i| > 0$. The trivial solution $x(k) = 0$, $k = -m, -m+1, \dots$ of (1) is called Lyapunov exponentially stable if there exist constants $N > 0$ and $\theta \in (0, 1)$ such that, for an arbitrary solution $x = x(k)$ of (1),

$$|x(k)| \le N \|x(0)\|_m \theta^k, \;\; k = 1, 2, \dots$$

where $\|x\,(0)\|_m := \max\{|x\,(i)|, i = -m, -m+1, \dots, 0\}$. As it is customary, the asymptotic stability of (1) can be investigated by analyzing the roots of the related characteristic equation. The characteristic equation relevant to (1) is a polynomial equation of degree $(m+1)\,n$. For large m and n, it is impossible, in a general case, to solve such a problem. For example, the Schur-Cohn criterion is not applied because the computer calculation is too time-consuming. The exponential stability of (1) is analyzed by the second Lyapunov method and the following well known result is utilized: If $\rho(A) < 1$, then the Lyapunov matrix equation

$$A^T HA - H = -C \tag{2}$$

has a unique solution - a positive definite symmetric matrix H for an arbitrary positive definite symmetric $n \times n$ matrix C. Let $\gamma > 1$ be a parameter. Define auxiliary numbers

$$\begin{aligned}
L_1 :=& \gamma \left[\lambda_{\max}(H) - \lambda_{\min}(C) + \sum_{i=1}^{s} |A^T H B_i| \right], \\
L_2 :=& \lambda_{\min}(H) - \frac{1}{2}\gamma\varphi(H) \left[2\sum_{i=1}^{s} \gamma^{m_i} |A^T H B_i| + \sum_{i,j=1}^{s} \left[\gamma^{m_i+1} + \gamma^{m_j+1}\right] |B_i^T H B_j| \right], \\
L_3 :=& \lambda_{\min}(C) - \sum_{i=1}^{s} |A^T H B_i| - \frac{\gamma - 1}{\gamma} \lambda_{\max}(H) \\
& - \frac{1}{2}\varphi^2(H) \left[2\sum_{i=1}^{s} \gamma^{m_i} |A^T H B_i| + \sum_{i,j=1}^{s} \left[\gamma^{m_i+1} + \gamma^{m_j+1}\right] |B_i^T H B_j| \right].
\end{aligned}$$

Theorem 1 *Let $\rho(A) < 1$, C be a fixed positive definite symmetric $n \times n$ matrix, let matrix H solve the equation (2), and, for a fixed $\gamma > 1$, let $L_1 > 0$, $L_2 > 0$, $L_3 \geq 0$. Then, the system (1) is exponentially stable and, for an arbitrary solution $x = x(k)$, the estimate*

$$|x(k)| \leq \sqrt{\varphi(H)}\, \|x(0)\|_m \, \gamma^{-k/2}, \quad k \geq 1$$

holds.