Viable trajectories for nonconvex differential inclusions with constant delay ⋆

Hélène Frankowska * **Ihab Haidar** **

* *CNRS, IMJ-PRG, Sorbonne Université, Case 247, 4 Place Jussieu, 75252 Paris, France (e-mail: helene.frankowska@imj-prg.fr)*
** *Quartz EA 7393, ENSEA, 6 Avenue du Ponceau 95014, Cergy-Pontoise cedex, France (e-mail: ihab.haidar@ensea.fr)*

Abstract: In this paper, we consider a nonconvex differential inclusion with constant delay. We study the existence of viable solutions when the state is constrained to the closure of an open subset of $\mathbb{R}^n$. The main contribution is a relaxation result stating that, under some assumptions, each "viable solution" of the convexified inclusion can be approximated by "viable solutions" of the original one. This result is obtained thanks to an extension of the celebrated Filippov's theorem to the case of delay differential inclusions.

Keywords: Delay differential inclusions, relaxation, state constraints, inward pointing conditions.

1. INTRODUCTION

Time delay systems are convenient to model some complex systems arising in population dynamics or engineering sciences. Delays appear naturally in the state variable or in the control when dealing with models involving control systems, and even in both control and state variables. Different approaches have been developed in the literature in order to study stability, controllability, observability and optimality problems for such systems (see, e.g., Fliess and Mounier (1998); Göllmann et al. (2009); Niculescu (2001)). Differential inclusions is a convenient tool to work with various types of control systems (see, e.g., Aubin and Frankowska (1990)). For instance, a closed loop system can be written as a differential inclusion where, at each state, the set-valued map is defined by the set of all possible feedback controls at this state. Also, differential inclusions are helpful to study control systems with uncertainties, where the set-valued map incorporates model errors. In the presence of state constraints, the investigation of such differential inclusions becomes very difficult and their analysis has occupied a considerable attention in the literature (see, e.g., Aubin (1991); Frankowska et al. (2016); Frankowska and Mazzola (2013)).
In this paper, we are concerned with the differential inclusion with constant delay

$$\begin{cases} \dot{x}(t) \in F(t, x(t), x(t-\tau)), & a.e.\ t \in [t_0, T], \\ x_{t_0} = \varphi, \end{cases} \tag{1}$$

and its convexified, or relaxed, form

$$\begin{cases} \dot{x}(t) \in \operatorname{co} F(t, x(t), x(t-\tau)), & a.e.\ t \in [t_0, T], \\ x_{t_0} = \varphi, \end{cases} \tag{2}$$

where $\operatorname{co} F(t, x(t), x(t-\tau))$ denotes the convex hull of $F(t, x(t), x(t-\tau))$, $x(t) \in \mathbb{R}^n$, represents the state at time t, $x(t-\tau)$ the τ-delayed state, $\tau > 0$, $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a set-valued map with non-empty closed images, $0 \leq t_0 \leq T$ and $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is the initial condition. Recall that $x_t : [-\tau, 0] \to \mathbb{R}^n$ is a standard notation for the history function defined by $x_t(\theta) = x(t+\theta)$, for $-\tau \leq \theta \leq 0$. In the above, $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions from $[-\tau, 0]$ into $\mathbb{R}^n$, with the usual norm. We restrict our attention to trajectories of (1) or (2) which are subject to the constraint

$$x(t) \in K \qquad \forall\, t \in [t_0, T], \tag{3}$$

where K is a closed subset of $\mathbb{R}^n$. Such trajectories are called *feasible* or *viable* trajectories.
The existence of viable trajectories for (1) is closely related to the convexity of the values of the set-valued map F. In fact, even in the absence of delays, nonconvex-valued differential inclusions may not have viable solutions, while it is not the case of their convexification (see, e.g., (Aubin, 1991, Example p. 89)). When the set-valued map F is convex-valued, thanks to the viability theory of Aubin (1991), the existence of viable trajectories is completely characterized by a necessary and sufficient condition linking the geometry of the constraint set K to the set-valued map F. In the absence of delays, and under some regularity assumptions on F, this condition is as follows

$$\forall\, t \in [0,T], \forall\, x \in K, \quad F(t,x) \cap T_K(x) \neq \emptyset, \tag{4}$$

where $T_K(x)$ is the contingent cone to K at x. If (4) holds true, then the constraint set K is called *viability domain.* We underline that conditions like (4) are also given for a general type of convex-valued functional differential inclusions (see Aubin (1991); Haddad (1981) for more details).
The convexity hypothesis on the set-valued map F is very restrictive for some mathematical models. In order to compensate the lack of convexity, stronger regularity on F and stronger tangential conditions are needed. In the absence of delays, many works were devoted to various *inward pointing conditions* allowing to approximate

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relaxed feasible trajectories by feasible trajectories and provide estimates on the distance of a given trajectory of unconstrained control system from the set of its feasible trajectories, see for instance Bettiol et al. (2010, 2012); Forcellini and Rampazzo (1999); Frankowska et al. (2016); Frankowska and Mazzola (2013); Frankowska and Rampazzo (2000); Frankowska and Vinter (2000). In the literature, these estimates have been referred to as neighboring feasible trajectory (NFT) estimates.

In general, for various practical examples, condition like (4) is not fulfilled. In this case, the constraint set K is not a viability domain and the largest viable subset of K (called *viability kernel*) is considered. In the absence of delay, viability algorithms allowing the computation of the viability kernel have been conceived for convex-valued differential inclusions (see, e.g., Frankowska and Quincampoix (1991); Saint-Pierre (1994)). Based on these algorithms, numerical methods have been developed (see, e.g., Rouquier et al. (2015)) and used to find viability kernels for numerous examples coming from different fields (see, e.g., Aubin et al. (2011); Haidar et al. (2017)). In order to develop similar numerical tools in the case of delay differential inclusions, viability algorithms and relaxation theorems under state constraints are crucial. The latter point is the purpose of this paper.

Let $\lambda > 0$. Consider the following inward pointing condition:

$$(IPC_{rel}^{\lambda}) \begin{cases} \forall\, t \in [0,T], \forall\, x \in \partial K, \forall\, y \in x + \tau\lambda B, \\ \forall\, v \in F(t,x,y) \text{ such that } \max\limits_{n \in N_K^1(x)} \langle n, v\rangle \geq 0, \\ \exists\, w \in \operatorname{Liminf}_{(s,z,\xi)\to(t,x,y)} \operatorname{co} F(s,z,\xi) \\ \text{satisfying } \max\limits_{n \in N_K^1(x)} \langle n, w - v\rangle < 0, \end{cases}$$

where Liminf denotes the Kuratowski lower set limit (see Aubin and Frankowska (1990)), $N_K^1(x) := N_K(x) \cap S^{n-1}$, S^{n-1} is the unit sphere and $N_K(x)$ denotes the Clarke normal cone to K at x (see Clarke (1990)). Assuming (IPC_{rel}^{λ}), we give a relaxation result stating that the set of feasible trajectories of (1) is dense in the set of relaxed feasible trajectories of (2). This is proved by using several preliminary results. The first one is an extension of the Filippov theorem from Filippov (1967) to delay differential inclusions, which is an essential step to construct feasible trajectories. Then, we provide NFT estimates on the distance of a given trajectory from the set of feasible trajectories.

The paper is organized as follows. Section 2 presents the list of notations, definitions and assumptions in use. In Section 3 we state our main results. The proofs can be found in Frankowska and Haidar (2017). The application of our relaxation theorems in optimal control is discussed in section 4. An example showing the applicability of our results is given in Section 5.

2. PRELIMINARIES

In this section we list the notations and the main assumptions in use.

2.1 Notations and definitions

Consider the Euclidean space $(\mathbb{R}^n, \|\cdot\|)$, where n is a positive integer. We denote by $B(x,r)$ the closed ball of center $x \in \mathbb{R}^n$ and radius $r > 0$ and by B the closed unit ball in $\mathbb{R}^n$ centered at 0. Let $\operatorname{co} A$ stands for the convex hull of a subset $A \subset \mathbb{R}^n$. Given interval $I \subset \mathbb{R}$, $(\mathcal{C}(I,\mathbb{R}^n), \|\cdot\|_C)$ denotes the Banach space of continuous functions from I into $\mathbb{R}^n$, where $\|\cdot\|_C$ is the norm of uniform convergence. We denote by $\mathcal{L}^1(I,\mathbb{R}^n)$ the space of Lebesgue integrable functions from I to $\mathbb{R}^n$. Let K be a nonempty closed subset of $\mathbb{R}^n$, $\operatorname{Int} K$ be its interior and ∂K its boundary, $d_K(x) = \inf_{y\in K} \|x - y\|$ is the distance from x to K.

We will use the following notion of solution:

Definition 1. Let $0 \leq t_0 \leq T$, $\tau > 0$ *and* $\varphi \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$. *A function* $x \in \mathcal{C}([t_0-\tau,T],\mathbb{R}^n)$ *is called an* F*-trajectory, if* $x(\cdot)$ *is absolutely continuous on* $[t_0,T]$ *and satisfies* (1).

An F-trajectory which verifies the state constraint (3) is called feasible F-trajectory. A trajectory associated to the relaxed differential inclusion (2) is called relaxed F-trajectory, and relaxed feasible F-trajectory if in addition (3) holds true.

2.2 Assumptions

Let $0 \leq t_0 \leq T$, $\tau > 0$ and $F : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a set-valued map with non-empty closed images. In our main theorems, we will assume the following regularity conditions on F:

(A1) for every $X = (x,y) \in \mathbb{R}^n\times\mathbb{R}^n$ the set-valued map $F(\cdot,X)$ is measurable;

(A2) the set-valued map $F(t,\cdot)$ is locally Lipschitz, i.e. $\forall\, R > 0$, $\exists\, \zeta_R(\cdot) \in \mathcal{L}^1([t_0,T],\mathbb{R}^+)$ such that, for a.e. $t \in [t_0,T]$ and any $X = (x_1,y_1), Y = (x_2,y_2) \in RB \times RB$
$$F(t,X) \subset F(t,Y) + \zeta_R(t)\|X - Y\|B;$$

(A3) the set-valued map F has a sublinear growth, i.e. there exists $\sigma > 0$ such that, for a.e. $t \in [t_0,T]$ and any $X = (x,y) \in \mathbb{R}^n\times\mathbb{R}^n$
$$F(t,X) \subset \sigma(1 + \|X\|)B;$$

(A4) for a given $\lambda > 0$, the set-valued map F is upper semicontinuous on $[t_0,T]\times\partial K\times(\partial K + \tau\lambda B)$ i.e. for all $t \in [t_0,T]$ and all $X \in \partial K\times(\partial K + \tau\lambda B)$, we have $F(t,X) \neq \emptyset$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$F(s,Y) \subset F(t,X) + \varepsilon B \quad \forall\,(s,Y) \in B((t,X),\delta).$$

3. MAIN RESULTS

3.1 Filippov's Theorem

The following theorem extends the celebrated Filippov's theorem from Filippov (1967), to differential inclusions of type (1).

Theorem 1. Let $\beta > 0$ *and* $\delta_0 \geq 0$ *and assume* (A1), (A2). *Let* $y \in \mathcal{C}([t_0-\tau,T],\mathbb{R}^n)$ *be such that* $y(\cdot)$ *is absolutely continuous on* $[t_0,T]$. *Set* $R = \max\limits_{t\in[t_0-\tau,T]} \|y(t)\|$,

$$\begin{aligned}\gamma_1(t) &= d_{F(t,y(t),y(t-\tau))}(\dot y(t)),\\ \gamma_2(t) &= \exp\left\{\int_{t_0}^{t}\zeta_{R+\beta}(s)ds\right\},\\ \gamma_3(t) &= \gamma_2(t)\left(\delta_0+\int_{t_0}^{t}\gamma_1(s)ds\right).\end{aligned}\tag{5}$$

If $\gamma_3(T) < \beta$, then for all $\varphi \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$ with $\|\varphi - y_{t_0}\|_C \le \delta_0$, there exists $x \in \mathcal{C}([t_0-\tau,T],\mathbb{R}^n)$ such that $x(\cdot)$ is an F-trajectory and for all $t \in [t_0,T]$

$$\|x_t - y_t\|_C \le \gamma_3(t)$$

and for almost every $t \in [t_0,T]$,

$$\|\dot x(t) - \dot y(t)\| \le \zeta_{R+\beta}(t)\gamma_3(t) + \gamma_1(t).$$

Sketch of proof: Starting from the reference trajectory $x_0 \equiv y$, we construct the sequences $x_n \in \mathcal{C}([t_0-\tau,T],\mathbb{R}^n)$ and $f_n \in \mathcal{L}^1([t_0,T],\mathbb{R}^n)$, *for* $n \ge 1$, such that

$$\begin{cases} x_n(t) = \varphi(0) + \displaystyle\int_{t_0}^{t} f_n(s)ds,\ t \in [t_0,T],\\ x_{n,t_0} = \varphi,\end{cases}$$

and

$$f_n(t) \in F(t, x_{n-1}(t), x_{n-1}(t-\tau)), \quad t \in [t_0,T],$$

with

$$\|f_{n+1}(t) - f_n(t)\| \le \zeta_{R+\beta}(t)\|x_{n,t} - x_{n-1,t}\|_C,$$

for almost every $t \in [t_0,T]$. Passing to the limits, we prove the existence of $x \in \mathcal{C}([t_0-\tau,T],\mathbb{R}^n)$, an F-trajectory which verifies Theorem 1. □

The following theorem establishes the possibility of approximating any relaxed F-trajectory by an F-trajectory starting from the same initial condition.

Theorem 2. Let $y(\cdot)$ be a relaxed F-trajectory. Assume (A1), (A2) and (A3). *Then for every $\delta > 0$ there exists an F-trajectory $x(\cdot)$* satisfying $x_{t_0} = y_{t_0}$ and $\sup_{t\in[t_0,T]}\|x(t)-y(t)\| \le \delta$.

The proof of Theorem 2 uses standard relaxation arguments which are adapted to differential inclusions with delay.

3.2 Neighboring feasible trajectories theorems

Let $\lambda > 0$. Consider the following inward pointing condition:

$$(IPC^{\lambda})\begin{cases} \forall\, t \in [0,T], \forall\, x \in \partial K, \forall\, y \in x + \tau\lambda B,\\ \forall\, v \in F(t,x,y) \text{ such that } \displaystyle\max_{n\in N_K^1(x)}\langle n, v\rangle \ge 0,\\ \exists\, w \in \mathrm{Liminf}_{(s,z,\xi)\to(t,x,y)}F(s,z,\xi)\\ \text{satisfying } \displaystyle\max_{n\in N_K^1(x)}\langle n, w - v\rangle < 0.\end{cases}$$

This condition is an extension of the well known inward pointing conditions to delay differential inclusions. It is equivalent, in the case of continuous set-valued map F and smooth boundary ∂K, to the following

$$\begin{cases} \forall\, t \in [0,T], \forall\, x \in \partial K, \forall\, y \in x + \tau\lambda B,\\ \exists\, w \in F(t,x,y) \text{ satisfying } \langle n_x, w\rangle < 0,\end{cases}$$

where n_x denotes the unit outward normal to K at x. Note that, in the case of non-smooth boundary, the condition $\max_{n\in N_K^1(x)}\langle n, w\rangle < 0$ is not sufficient in itself to construct feasible trajectories (see Bettiol et al. (2010) for a counter example). The (IPC^{λ}) condition can be interpreted as illustrated by figure 1: Let $t \in [0,T]$ and x be on the boundary of K. A velocity at x is a vector which depends particularly on its history. This history, denoted by y, lies in a ball of center x and radius $\tau\lambda$ (where τ is the constant delay and λ depends on F). This condition requires, for every velocity v pointing to the exterior of K, the existence of a vector w in the lower limit of $F(s,z,\xi)$ when $(s,z,\xi) \to (t,x,y)$ such that $w - v$ pointes into the interior of K.

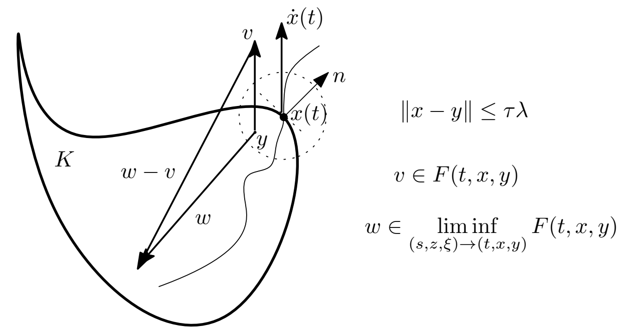

Fig. 1. Inward pointing condition.

The following theorem shows the existence of a feasible F-trajectory and provides an estimate of the distance (in the norm of uniform convergence) of this trajectory from a specified F-trajectory.

Theorem 3. Assume (A1)–(A3). *Let $\tau > 0$, $r_0 > 0$ and $\lambda_0 > 0$ and suppose that, for*

$$\lambda = \max\{\lambda_0, (1 + (1 + \lambda_0\tau + r_0)e^{\sigma T})\sigma\},\tag{6}$$

assumptions (A4) *and (IPC^{λ}) hold true. Then there exists a constant $C > 0$ such that for any $t_0 \in [0,T]$ and every F-trajectory $\hat x(\cdot)$ on $[t_0-\tau,T]$ with λ_0-Lipschitz $\hat x_{t_0}$ and $\hat x(t_0) \in K \cap r_0 B$, and for any $\varepsilon_0 > 0$, we can find a feasible F-trajectory $x(\cdot)$ on $[t_0-\tau,T]$ satisfying $x_{t_0} = \hat x_{t_0}$, $x((t_0,T]) \subset \mathrm{Int}\,K$ and*

$$\|x_t - \hat x_t\|_C \le C\left(\max_{t\in[t_0,T]} d_K(\hat x(t)) + \varepsilon_0\right).\tag{7}$$

Sketch of proof: The proof of this theorem is done in two main steps. Firstly, we prove the existence of solutions which satisfy the state constraint only on a subinterval of $[t_0,T]$. We show the existence of positive constants δ and c for which, for every $\bar t \in [0,T]$ and every F-trajectory $\hat x(\cdot)$ on $[\bar t-\tau,T]$ and any $\varepsilon > 0$, we can find an F-trajectory on $[\bar t-\tau,T]$ satisfying

$$\begin{cases} x_{\bar t} = \hat x_{\bar t},\\ x(t) \in \mathrm{Int}\,K, \qquad \forall\, t \in (\bar t, (\bar t+\delta)\wedge T]\\ \|x_t - \hat x_t\|_C \le c\displaystyle\max_{t\in[\bar t,T]} d_K(\hat x(t)) + \varepsilon.\end{cases}\tag{8}$$

Knowing that δ and c are independent from $\bar t$ and $\hat x(\cdot)$, the second step consists to reproduce recursively the property given by (8), with $\bar t = (t_0 + i\delta)\wedge T$ and $\hat x_{\bar t} = x_{\bar t}$, until achieving $(t_0 + i\delta)\wedge T = T$, for some $i \ge 0$. This leads to the construction of a finite sequence of arcs whose concatenations is an F-trajectory satisfying the state constraint on $[t_0,T]$ together with the neighboring feasible trajectory estimation (7). □

Theorem 3 together with Theorem 2 imply that under the inward pointing condition (IPC^{λ}), the set of F-trajectories lying in the interior of the constraint set K,

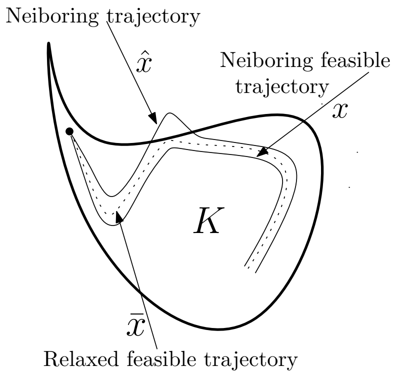

Fig. 2. Relaxation theorem under state constraints.

for $t \in (t_0, T]$ and starting at $\hat{x}_{t_0}$, is dense in the set of feasible relaxed F-trajectories. This results from the following corollary:

Corollary 4. Under all the assumptions of Theorem 3, for any feasible relaxed F-trajectory $\bar{x}(\cdot)$ with λ_0-Lipschitz $\bar{x}_{t_0}$ and $\bar{x}(t_0) \in K \cap r_0 B$, and any $\delta > 0$, there exists a feasible F-trajectory $x(\cdot)$ such that $x_{t_0} = \bar{x}_{t_0}$, $x((t_0, T]) \in \operatorname{Int} K$ and $\|x_t - \bar{x}_t\|_C < \delta$ for all $t \in [t_0, T]$.

Sketch of proof: The proof of this corollary is illustrated by figure 2. Let $\delta > 0$. Starting from a feasible relaxed trajectory $\bar{x}$, thanks to Theorem 2, we can find an F-trajectory $\hat{x}$ (which may violate the constraint set) such that $\|\bar{x}_t - \hat{x}_t\|_C < \delta/2$ for all $t \in [t_0, T]$. For the neighboring trajectory $\hat{x}$, thanks to Theorem 3, we can find a feasible F-trajectory x, such that $\|\hat{x}_t - x_t\|_C < \delta/2$ for all $t \in [t_0, T]$. Then, $\|x_t - \bar{x}_t\|_C < \delta$, for all $t \in [t_0, T]$. $\square$

Now, assume the relaxed inward pointing condition given by (IPC^{λ}_{rel}). The following theorem is related to Theorem 3, however neither one is contained in another.

Theorem 5. Assume (A1)–(A3)*. Let $\tau > 0$, $r_0 > 0$ and $\lambda_0 > 0$ and suppose that, for λ given by* (6)*, assumptions* (A4) *and (IPC^{λ}_{rel}) hold true. Then there exists a constant $C > 0$ such that for any $t_0 \in [0, T]$ and every relaxed F-trajectory $\hat{x}(\cdot)$ on $[t_0 - \tau, T]$ with λ_0-Lipschitz $\hat{x}_{t_0}$ and $\hat{x}(t_0) \in K \cap r_0 B$, and for any $\varepsilon_0 > 0$, we can find a relaxed feasible F-trajectory $x(\cdot)$ on $[t_0 - \tau, T]$ satisfying $x_{t_0} = \hat{x}_{t_0}$, $x((t_0, T]) \subset \operatorname{Int} K$ and*

$$\|x_t - \hat{x}_t\|_C \le C \left(\max_{t \in [t_0, T]} d_K(\hat{x}(t)) + \varepsilon_0 \right).$$

Theorem 5 and the constructive argument of (Bettiol et al., 2012, Proof of Lemma 5.2) imply the following Corollary:

Corollary 6. Under all the assumptions of Theorem 5, for any relaxed feasible F-trajectory $\bar{x}(\cdot)$ with λ_0-Lipschitz $\bar{x}_{t_0}$ and $\bar{x}(t_0) \in K \cap r_0 B$, and any $\delta > 0$, there exists a feasible F-trajectory $x(\cdot)$ such that $x_{t_0} = \bar{x}_{t_0}$, $x((t_0, T]) \in \operatorname{Int} K$ and $\|x_t - \bar{x}_t\|_C < \delta$ for all $t \in [t_0, T]$.

The proofs of Theorem 5 and Corollary 6 are the adaptation of what is done before to the case of the relaxed inward pointing condition (IPC^{λ}_{rel}).

4. APPLICATIONS OF THE RELAXATION THEOREMS IN OPTIMAL CONTROL

The relaxation theorems obtained in this paper allow to show that the value function of the original optimal control problem coincides with the value function of the relaxed one. This is described, briefly, by the following.
Let $g : \mathbb{R}^n \to \mathbb{R}$ be a λ_1-Lipschitz function. Suppose that the assumptions of Theorem 5 are satisfied. Let $\mathcal{S}^{\mathscr{K}_\lambda}_{[t_0,T]}(x_0)$ be the set of all solutions to (1), (3), where $x_{t_0} = x_0$ and $x_0 \in \mathscr{K}_\lambda$, with

$$\mathscr{K}_\lambda := \{\psi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n) : \psi \text{ is } \lambda\text{-Lipschitz}, \psi(0) \in K\}.$$

Consider the Mayer optimal control problem

$$\min\{g(x(T)) : x(\cdot) \in \mathcal{S}^{\mathscr{K}_\lambda}_{[0,T]}(x_0)\}. \tag{9}$$

The value function, associated to problem (9),

$$V : [0, T] \times \mathscr{K}_\lambda \to \mathbb{R} \cup \{+\infty\},$$

is defined by

$$V(t_0, y_0) = \inf\{g(x(T)) : x(\cdot) \in \mathcal{S}^{\mathscr{K}_\lambda}_{[t_0,T]}(y_0)\} \tag{10}$$

with the convention that $V(t_0, y_0) = +\infty$ if $\mathcal{S}^{\mathscr{K}_\lambda}_{[t_0,T]}(y_0) = \emptyset$. Thanks to Corollary 6, one can prove that V is equal to the value function of the relaxed Mayer problem, and thus any optimal solution to the Mayer problem is also optimal for the relaxed Mayer problem. Indeed, let us denote by $\overline{V}$ the value function of the relaxed Mayer problem. Fix $(t_0, y_0) \in [0, T] \times \mathscr{K}_\lambda$. We have clearly $\overline{V}(t_0, y_0) \le V(t_0, y_0)$. On other hand, for every $\varepsilon > 0$,

$$\begin{aligned} V(t_0, y_0) &\le g(x(T)) \\ &\le g(\bar{x}(T)) + \lambda_1 \|x(T) - \bar{x}(T)\| \\ &\le \overline{V}(t_0, y_0) + \varepsilon, \end{aligned} \tag{11}$$

where $\bar{x}(\cdot)$ is a relaxed feasible trajectory verifying $\overline{V}(t_0, y_0) \ge g(\bar{x}(T)) - \varepsilon/2$ and $x(\cdot)$ an associated feasible trajectory satisfying (thanks to Corollary 6)

$$\|x_t - \bar{x}_t\|_{\mathcal{C}} \le \varepsilon/2\lambda_1, \quad \forall t \in [0, T]. \tag{12}$$

Being true for arbitrarily small ε, inequality (11) implies that $\overline{V}(t_0, y_0) = V(t_0, y_0)$.
In addition, Theorem 2 together with Theorem 5, allow to prove that V is Lipschitz on $\mathscr{K}_\lambda$. This latter property allows to characterize the optimal solutions of the Mayer problem by means of the relaxed differential inclusion (see (Frankowska and Mazzola, 2013, Theorem 5.3), for more details in the case of differential inclusions without delay).

5. EXAMPLE: LANDFILL WASTE MANAGEMENT

Here, we present an example which concerns the management of landfill waste. A landfill is a controlled site for the disposal of waste materials where the solid waste is disposed and treated. The treatment is decomposed in two parts: “biological treatment” of the solubilized substrates by means of a microbial biodegradation and “physical treatment” of the unsolubilized substrates through the re-circulation of the landfill leachate. In fact, landfill leachate is the liquid that drains from waste during the landfill operation; knowing its highest acidity, the re-circulation of this waste-water improve the system mixing and then the solubilization of the solid material. A simplified model of this complex system is proposed in Rapaport et al. (2016). This is given by [1]

$$\begin{aligned} \dot{x}_1 &= -f(x_1, u) \\ \dot{x}_2 &= f(x_1, u) - g(x_2)x_3, \\ \dot{x}_3 &= g(x_2)x_3, \end{aligned} \tag{13}$$

where x_1 and x_2 denote the unsolubilized and solubilized substrates concentrations, x_3 is the concentration of the biomass that degrades the solubilized substrate and g is the biomass growth rate function. The function f describes the transfer kinetics of x_1 to x_2, which is parametrized by u, the leachate re-circulation flow; it is supposed to be positive, equal to zero at $x_1 = 0$ ($u = 0$, respectively) and increasing with respect to x_1 (u, respectively). The function g describes the specific microbial growth rate, it is usually modeled by the following Haldane function (see, e. g., Smith and Waltman (1995))

$$g(x_2) = \frac{\bar{g}x_2}{k_1 + x_2 + x_2^2/k_2},$$

where $\bar{g}$, k_1 and k_2 are positive parameters. The objective is to drive the total concentration of the substrate below a fixed threshold. The leachate re-circulation flow is the control which takes values in $[0, u_{max}]$. The state constraints are given by

$$K = \left\{(x_1, x_2, x_3) \in \mathbb{R}^+ : 0 \leq x_1 + x_2 \leq M\right\}, \tag{14}$$

where $M > 0$ is a fixed threshold on the total substrate concentration.

In order to be more realistic, a time-delay should be incorporated in (13). This delay is due to the process of bioconversion from substrate degradation to biomass growth. It may also represents the lag phase in the growth response of microorganisms in a fluctuant environment (see, e.g., MacDonald (1976) for more details). The effects of delayed growth response on the dynamic behaviors of models like (13) have been widely studied in the literature (see, e.g, Meng et al. (2010), where we show how the consideration of such delays change drastically its dynamic behaviors). Thus, it is judicious to consider such a delay in the growth response of (13). Let $\tau > 0$, this can be formulated by the following equations

$$\begin{aligned} \dot{x}_1 &= -f(x_1, u) \\ \dot{x}_2 &= f(x_1, u) - g(x_2)x_3, \\ \dot{x}_3 &= g(x_2(t-\tau))x_3(t-\tau). \end{aligned} \tag{15}$$

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. System (15) can be written in the form of (1), where the set-valued map $F : \mathbb{R}^3 \times \mathbb{R}^3 \rightsquigarrow \mathbb{R}^3$ is given by

$$F(x, y) = \bigcup_{u \in U} \begin{pmatrix} f(x_1, u) \\ f(x_1, u) - g(x_2)x_3 \\ g(y_2)y_3 \end{pmatrix}.$$

In general, F is not convex-valued. If $f(\cdot)$ and $g(\cdot)$ are sufficiently regular then F fulfills the assumption of Theorem 5. Thus, if condition (IPC^{λ}_{rel}) holds true on the boundary of K, Corollary 6 guaranties the existence of feasible trajectories for (15), approximating feasible relaxed trajectories of the convexified problem.

[1] By abuse of notation, we omit writing explicitly the dependence of $\dot{x}_i, x_i$, and u on t, for $i = 1, 2, 3$.

6. CONCLUSION

In this paper we deal with an interesting problem concerning the existence of solutions for delay differential inclusions, in presence of state constraints. We present a relaxation result stating that the set of trajectories lying in the interior of the constraint is dense in the set of constrained trajectories of the corresponding convexified inclusion. The proof of this result is given in several steps. First, in Theorem 1, we extend the celebrated Filippov's theorem to differential inclusions with constant delay. Then, thanks to this extension, we give, in Theorem 2, a relaxation result without state constraint. Then, we generalize this latter result to the case where the state variable is constrained to the closure of an open subset of $\mathbb{R}^n$. Under the new inward pointing condition IPC^{λ}, we show, in Corollary 4, that each feasible trajectory of a delay differential inclusion can be approximated (in the norm of uniform convergence) by trajectories lying in the interior of the constraint set K and starting at the same initial condition. We also estimate, in Theorem 3, the distance between feasible and non-feasible trajectories, using the magnitude of the constraints violation. This result is refined in Corollary 6 by considering the relaxed inward pointing condition IPC^{λ}_{rel}. Thanks to our relaxation theorems, we show that the value function of an optimal control problem coincides with the value function of its relaxed one. An example illustrating the applicability of our result is also discussed.

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