# Global Stabilization of a Class of Time-Delay Nonlinear Systems with Unknown Control Directions by Nonsmooth Feedback ⋆

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**Abstract:** This paper addresses the problem of global stabilization for a class of time-delay systems with inherent nonlinearity and unknown control directions, which may not be controlled by any smooth feedback. The dynamic gain based approach in Zhang, Lin and Lin (2017) and the idea of Nussbaum-gain function Nussbaum (1983) are employed to deal with inherently nonlinear systems with time-delay in the presence of unknown control directions. With the help of appropriate Lyapunov-Krasovskii functionals, we design a non-smooth delay-independent feedback control law that guarantees the global asymptotic convergence of the system state and global boundedness of the resulting closed-loop system.

## 1. INTRODUCTION

Control of time-delay nonlinear systems is an important yet challenging problem. It is often encountered in many real-world applications that involve time-delay, such as network control, mechanical systems, biological systems and chemical processes, etc. In this work, we first consider the following class of time-delay nonlinear systems with unknown control directions

$$\begin{aligned} \dot{x}_i &= \theta_i x_{i+1}^{p_i} + f_i(x_1, \cdots, x_i, x_1(t-d), \cdots, x_i(t-d)), \\ \dot{x}_n &= \theta_n u^{p_n} + f_n(x, x(t-d)), \\ x(s) &= \zeta(s), s \in [-d, 0], \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state and input, respectively. The constant $d \geq 0$ is an unknown time-delay of the system, $p_i > 0$ are odd integers, $f_i : \mathbb{R}^{2i} \to \mathbb{R}$ are $C^1$ mappings with $f_i(0,0) = 0$, and $\zeta(s) \in \mathbb{R}^n$ is a continuous function defined on $[-d, 0]$. The coefficients $\theta_i \neq 0, 1 \leq i \leq n$, are unknown constants bounded by a known constant $\bar{c}$.

For the analysis and synthesis of time-delay systems Gu, Kharitonov and Chen (2003); Jankovic (2001); Richard (2003), the Lyapunov-Krasovskii and Lyapunov-Razumikhin methods are two powerful and common tools that have been found wide applications. In the literature, research of time-delay systems can be classified primarily into three different categories. The first category of study focuses on the time-delay in the system state Gu, Kharitonov and Chen (2003), while the second one is aimed at the time-delay in the control input Mazenc, Modie and Niculescu (2003); Mazenc, Mondie and Francisco (2003); Liberis and Krstic (2013); Zhang, Boukas, Lui and Baron (2010). The last category addresses a general case where the time-delay is present in both the control input and the system state. For each category of time-delay nonlinear control problems, substantial progress has been made and various results have been obtained; see, for instance, Mazenc, Modie and Niculescu (2003); Mazenc, Mondie and Francisco (2003); Zhang, Zhang and Lin (2014); Zhang, Lin and Lin (2017) and references therein.

In the the case when $\theta_i = 1$, the time-delay nonlinear system (1) is in general not stabilizable, even locally, by any smooth state feedback. For instance, the time-delay system $\dot{x} = u^3 + x(t-d)$ cannot be stabilized by any smooth state feedback even when $d = 0$, and hence it is impossible to be smoothly stabilizable for $d \geq 0$. However, it is easy to verify that with the aid of the Lyapunov-Krasovskii functional $V = x^2 + \int_{t-d}^{t} x^2(s)ds$, the system is globally stabilizable by non-smooth but $C^0$ feedback $u = -(2x)^{1/3}$. Motivated by this observation and the under-actuated mechanical system Qian and Lin (2001b) in the presence of time-delay, we focus in this paper on the problem how to control the time-delay nonlinear system (1) by *non-smooth,* rather than smooth, delay-independent state feedback.

Most of the results obtained so far have been concentrated on time-delay nonlinear systems with known control directions, e.g., the signs of all coefficients of the chain of integrator are assumed to be known. If this crucial information is not available, a new method needs to be developed for the control of time-delay nonlinear systems. Since the sign of the control input often represents, for instance, motion directions of mechanical systems (such as robotics modeled by the Lagrange equation) and may be unknown, it is certainly interesting to investigate how to control time delay systems with unknown control directions. For the time-delay system with unknown control direction (1), global stabilization by delay-independent state feedback is not a trivial problem. The difficulties are i) when the signs of coefficients of a chain of integrators are unknown, the design of virtual controllers is less intuitive and more involved as the uncertainties cannot be cancelled directly by a conventional backstepping design; ii) the presence of time delay nonlinearities makes a delay-free, static state feedback law insufficient for mitigating the effects of time-delay, and hence a *dynamic* instead of static state feedback seem to be necessary.

Motivated by the universal control idea Nussbaum (1983); Lei and Lin (2006, 2007) and the recent development Zhang, Lin and Lin (2017), we propose a novel method for the construction of a set of Lyapunov-Krasovskii functionals and a delay-free, dynamic state feedback control scheme for counteracting the effects of time-delay nonlinearities and unknown control directions in the system (1) simultaneously. With the help of the new dynamic gain-based

⋆ This work was supported in part by the Thailand Research Fund under Grant RSA6080027 and Royal Golden Jubilee Ph.D. Program (Grant No. PHD/0158/2552), and by the Key Project of NSFC under Grants 61533009, 111 Project (B08015), Research Projects (JCY20150403161923519) and (KCYKYQD2017005). Corresponding author: Wei Lin (linwei@case.edu)

Lyapunov-Krasovskii functionals, we are able to design a delay free, dynamic state feedback compensator step-by-step, resulting in a solution to the global state regulation of the time-delay system (1) with boundedness. Interestingly, it is worth pointing out that the approach presented in this paper also provides a new yet simpler way for the design a dynamic state compensator that achieves global stabilization of the time-delay nonlinear system (1), in the absence of unknown control direction.

*Notations*: Denote $\bar{v}_i = [v_1, \cdots, v_i]^T \in \mathbb{R}^i$, for $i = 1, \cdots, n$. For instance, $\bar{x}_i = [x_1, \cdots, x_i]^T$, $\bar{x}_i(t-d) = [x_1(t-d), \cdots, x_i(t-d)]^T$ and $\bar{l}_i = [l_1, \cdots, l_i]^T$. A Nussbaum function $N(k) = k^2 \cos(k)$, which is obviously an even function, will be used in this work. It is not difficult to verify the following properties: 1) $\lim_{k\to+\infty} \sup \frac{1}{k}\int_0^k N(s)ds = +\infty$; 2) $\lim_{k\to+\infty} \inf \frac{1}{k} \int_0^k N(s)ds = -\infty$.

## 2. PRELIMINARY

To design a non-smooth but $C^0$ state feedback controller for the time-delay system (1), we introduce several key lemmas to be used throughout this paper.

*Lemma 1.* Qian and Lin (2001a,b) For positive real numbers $m, n$ and a real-valued function $\pi(x, y) > 0$, the following inequality holds $\forall x, y \in \mathbb{R}$.

$$|x|^m |y|^n \le \frac{m}{m+n}\pi(x,y)|x|^{m+n} + \frac{n}{m+n}\pi^{-m/n}(x,y)|y|^{m+n} \tag{2}$$

*Lemma 2.* Lin and Qian (2002) For a $C^0$ function $f(x,y)$, $\exists$ smooth functions $a(x) \ge 0$, $b(y) \ge 0$, $c(x) \ge 1$ and $d(y) \ge 1$, such that

$$|f(x,y)| \le a(x) + b(y), \quad |f(x,y)| \le c(x)\, d(y). \tag{3}$$

*Lemma 3.* Qian and Lin (2001a,b) Let $x, y \in \mathbb{R}$ and $p \ge 1$ be an integer. Then,

$$\begin{aligned} &|x+y|^p \le 2^{p-1}|x^p + y^p|, \\ &(|x| + |y|)^{\frac{1}{p}} \le |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \le 2^{\frac{p-1}{p}}(|x| + |y|)^{\frac{1}{p}}. \end{aligned} \tag{4}$$

If $p$ is an odd positive integer, then

$$|x-y|^p \le 2^{p-1}|x^p - y^p|. \tag{5}$$

*Lemma 4.* Zhang, Lin and Lin (2017) For a $C^0$ function $f(x,y)$ and a positive integer $k$, there exist smooth functions $g(x) \ge 0$ and $h(y) \ge 0$, such that

$$f(x,y)(|x|^k + |y|^k) \le g(x)|x|^k + h(y)|y|^k. \tag{6}$$

*Lemma 5.* Zhang, Lin and Lin (2017) For the $C^1$ function $f_i(\bar{x}_i, \bar{x}_i(t-d))$ with $f_i(0,0) = 0$, there exist smooth functions $\bar{\gamma}_{ij}(x_j) \ge 0$ and $\bar{\gamma}^*_{ij}(x_j(t-d)) \ge 0, j = 1, \cdots, i$, such that

$$|f_i(\cdot)| \le \Sigma_{j=1}^i (\bar{\gamma}_{ij}(x_j)|x_j| + \bar{\gamma}^*_{ij}(x_j(t-d))|x_j(t-d)|) \tag{7}$$

## 3. NONSMOOTH FEEDBACK WITH DYNAMIC GAINS

In this section, we utilize the idea from universal control Nussbaum (1983); Lei and Lin (2006, 2007), coupled with the feedback control strategy in Zhang, Lin and Lin (2017), to design a delay-free, dynamic state compensator that achieves global asymptotic state regulation with boundedness for the time-delay nonlinear system (1), As we shall see, the proposed dynamic compensator is composed of two sets of dynamic state feedback controllers. One of them is capable of mitigating the effects of the unknown control direction, while the other one is able to counteract the time-delay nonlinearities of the system (1). Notably, the idea of utilizing two sets of gain update laws has been explored in the area of adaptive control of nonlinear systems with unknown parameters by output feedback Lei and Lin (2006, 2007). This paper further demonstrates how a similar philosophy can be applied to effectively control the time-delay system (1) with unknown control direction.

*Theorem 6.* For the time-delay nonlinear system (1) whose control directions are not known, there exists a delay-free, dynamic state feedback controller of the form

$$\dot{L} = \eta(L,k,x), \quad \dot{k} = h(L,k,x), \quad u = \alpha(L,k,x) \tag{8}$$

with $\alpha(L,k,0) = 0$, such that the system state $x$ converges to the origin, while maintaining boundedness of the closed-loop system, where $\eta : \mathbb{R}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n-1}$, $h : \mathbb{R}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\alpha : \mathbb{R}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are $C^0$ mappings. □

**Proof.** We apply the adding a power integrator technique Qian and Lin (2001a,b), together with the idea of utilizing the Nussbaum functions Nussbaum (1983) and dynamic gains Zhang, Lin and Lin (2017); Lei and Lin (2006, 2007), to design a delay-free, non-smooth dynamic state compensator (8) that does the job.

**Step 1:** For the $x_1$—subsystem of the time-delay system (1) with the unknown sign of $\theta_1$, one can regard $x_2$ as a virtual control. Define $\xi_1 = x_1$ and construct the Lyapunov function $V_1(x_1, l_1) = \frac{1}{2}(1 + \frac{1}{l_1})\xi_1^2$, where $l_1(\cdot) \ge 1$ is a dynamic gain to be designed in Step 2. Then, a direct computation gives

$$\dot{V}_1 \le (1+\frac{1}{l_1})\theta_1\xi_1 x_2^{*p_1} - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 + 2\bar{c}|\xi_1\xi_2| + 2|\xi_1 f_1(x_1, x_1(t-d))|, \tag{9}$$

where $\xi_2 = x_2^{p_1} - x_2^{*p_1}$. In view of Lemma 2.5, we have $|f_1(\cdot)| \le \bar{\gamma}_1(x_1)|x_1| + \bar{\gamma}_1^*(x_1(t-d))|x_1(t-d)|$, for some smooth functions $\bar{\gamma}_1(\cdot) \ge 0$ and $\bar{\gamma}_1^*(\cdot) \ge 0$. Hence,

$$2|\xi_1 f_1(\cdot)| \le 2\xi_1^2\bar{\gamma}_1(x_1) + \xi_1^2 + \xi_1^2(t-d)\bar{\gamma}_1^{*2}(x_1(t-d)) \tag{10}$$

Use the bound $\bar{\gamma}_1^*(\cdot)$ to construct the Lyapunov-Krasovskii functional

$$V_{1LK} = V_1(x_1, l_1) + \int_{t-d}^{t} \xi_1^2(s)\bar{\gamma}_1^{*2}(x_1(s))\,ds.$$

From (9)-(10), it follows that

$$\dot{V}_{1LK} \le -n\xi_1^2 + [1+\frac{1}{l_1}]\theta_1\xi_1 x_2^{*p_1} - \frac{\dot{l}_1\xi_1^2}{2l_1^2} + \xi_1^2[2+n+2\bar{\gamma}_1(\cdot)+\bar{\gamma}_1^{*2}(\cdot)] + c_2\xi_2^2 \tag{11}$$

To cope with the unknown sign of $\theta_1$, we use the Nussbaum function Nussbaum (1983) for the design of a virtual controller. Specifically, a virtual controller with the Nussbaum gain can be constructed as

$$\begin{aligned} x_2^{*p_1} &= \xi_1 N(k_1)(2+n+2\bar{\gamma}_1(\cdot) + \gamma_1^{*2}(\cdot)) \quad := \xi_1 N(k_1)\beta_1(x_1) \\ \dot{k}_1 &= (1+\frac{1}{l_1})\xi_1^2\beta_1(x_1). \end{aligned} \tag{12}$$

This, together with $l_1(\cdot) \ge 1$, results in

$$\dot{V}_{1LK} \le -n\xi_1^2 + (\theta_1 N(k_1) + 1)\dot{k}_1 + c_2\xi_2^2 - \frac{\dot{l}_1}{2l_1^2}\xi_2^2. \tag{13}$$

**Step 2:** For the $(x_1, x_2)$—subsystem of the time-delay system (1) with the unknown sign of $\theta_2$, we construct the Lyapunov-Krasovskii functional

$$\begin{aligned} V_2 &= V_{1LK} + \frac{1}{l_1}k_1^2 W_2(\cdot) + \frac{1}{l_1 l_2}\left[\frac{\xi_1^2}{2} + k_1^2 W_2(\cdot)\right] \\ W_2(k_1, x_1, x_2) &= \int_{x_2^*}^{x_2} (s^{p_1} - x_2^{*p_1})^{2-1/p_1} ds, \end{aligned} \tag{14}$$

where $l_2(\cdot) \ge 1$ is a dynamic gain to be designed in the next step.

Following the argument in Qian and Lin (2001a,b), one can prove that $W_2(k_1, x_1, x_2)$ is $C^1$ and its partial derivatives are

$$\begin{aligned} \frac{\partial W_2}{\partial x_2} &= \xi_2^{2-1/p_1}, \\ \frac{\partial W_2}{\partial x_1} &= -\Big(2 - \frac{1}{p_1}\Big)\frac{\partial x_2^{*p_1}}{\partial x_1}\int_{x_2^*}^{x_2}(s^{p_1} - x_2^{*p_1})^{1-1/p_1} ds \\ \frac{\partial W_2}{\partial k_1} &= -\Big(2 - \frac{1}{p_1}\Big)\frac{\partial x_2^{*p_1}}{\partial k_1}\int_{x_2^*}^{x_2}(s^{p_1} - x_2^{*p_1})^{1-1/p_1} ds. \end{aligned} \tag{15}$$

Moreover, $m_2(x_2 - x_2^*)^{2p_1} \le W_2(\cdot) \le (2^{p_1} - 1)\xi_2^2$, for a constant $m_2 > 0$.

Since $l_j \geq 1$, it is deduced from (13) and (15) that

$$\dot{V}_2 \leq -n\xi_1^2 + (\theta_1 N(k_1)+1)\dot{k}_1 + c_2\xi_2^2 - \frac{\dot{l}_1}{2l_1^2}\xi_2^2 + \frac{k_1^2}{l_1}(1+\frac{1}{l_2})\theta_2\xi_2^{2-1/p_1}$$
$$\cdot(x_3^{*p_2} + x_3^{p_2} - x_3^{*p_2}) + \frac{2}{l_1}\left|k_1^2\left[\xi_2^{2-1/p_1}f_2(\cdot) + \frac{\partial W_2}{\partial x_1}\dot{x}_1 + \frac{\partial W_2}{\partial k_1}\dot{k}_1\right]\right.$$
$$\left.+k_1\dot{k}_1 W_2(\cdot)\right| + \frac{\xi_1\dot{x}_1}{l_1 l_2} - \frac{\dot{l}_1}{l_1^2}k_1^2 W_2(\cdot) - \frac{\dot{l}_1 l_2 + l_1\dot{l}_2}{l_1^2 l_2^2}(\frac{\xi_1^2}{2} + k_1^2 W_2(\cdot)) \quad (16)$$

From $\xi_2 = x_2^{p_1} - x_2^{*p_1}$, (12) and (15), it is not difficult to obtain (by Lemma 2.1 and Lemmas 2.3-2.5),

$$\frac{2k_1^2}{l_1}|\xi_2^{2-1/p_1}f_2(\cdot)| \leq k_1^2\xi_2^2\Upsilon_{21}(k_1,x_1,x_2) + \frac{1}{l_1}\xi_1^2\Upsilon_{22}(k_1,x_1)$$
$$+\frac{1}{l_1}\xi_1^2(t-d)\Upsilon_{22}^*(k_1(t-d),x_1(t-d))$$
$$+\xi_2^2(t-d)\Upsilon_{21}^*(k_1(t-d),x_1(t-d),x_2(t-d)), \quad (17)$$
$$\frac{2k_1^2}{l_1}\left|\frac{\partial W_2}{\partial x_1}\dot{x}_1 + \frac{\partial W_2}{\partial k_1}\dot{k}_1\right| + \frac{2}{l_1}k_1\dot{k}_1 W_2(\cdot) + \frac{1}{l_1 l_2}\xi_1\dot{x}_1$$
$$\leq k_1^2\xi_2^2\Phi_{21}(k_1,x_1,x_2) + \frac{1}{l_1}\xi_1^2\Phi_{22}(k_1,x_1) + \frac{1}{l_1}\xi_1^2(t-d)\Phi_2^*(x_1(t-d)),$$

where $\Upsilon_{2j}(\cdot) \geq 0$, $\Upsilon_{2j}^*(\cdot) \geq 0$, $\Phi_{2j}(\cdot) \geq 0$ and $\Phi_2^*(\cdot) \geq 0$, $j = 1,2$, are smooth functions. Using the bounds $\Upsilon_{2j}^*(\cdot)$ and $\Phi_2^*(\cdot)$ thus obtained, one can construct the Lyapunov-Krasovskii functional

$$V_{2LK} = V_2 + \int_{t-d}^{t}\xi_2^2(s)\Upsilon_{21}^*(k_1(s),x_1(s),x_2(s))ds$$
$$+\int_{t-d}^{t}\frac{1}{l_1(s)}\xi_1^2(s)[\Upsilon_{22}^*(k_1(s),x_1(s)) + \Phi_2^*(x_1(s))]ds \quad (18)$$

Then, it is deduced from (16) and (18) that

$$\dot{V}_{2LK} \leq -n\xi_1^2 - (n-1)k_1^2\xi_2^2 + (\theta_1 N(k_1)+1)\dot{k}_1 - \frac{\dot{l}_1}{2l_1^2}\xi_2^2 + \frac{1}{l_1}\xi_1^2\Big[\Phi_2^*(x_1)$$
$$+\Upsilon_{22}(k_1,x_1) + \Upsilon_{22}^*(k_1,x_1) + \Phi_{22}(k_1,x_1)\Big] + \frac{k_1^2}{l_1}(1+\frac{1}{l_2})\theta_2\xi_2^{2-1/p_1}x_3^{*p_2}$$
$$+\frac{2\bar{c}}{l_1}k_1^2|\xi_2^{2-1/p_1}(x_3^{p_2} - x_3^{*p_2})| + k_1^2\xi_2^2\Big[c_2 + (n-1) + \Upsilon_{21}(k_1,x_1,x_2)$$
$$+\Upsilon_{21}^*(k_1,x_1,x_2) + \Phi_{21}(k_1,x_1,x_2)\Big] - \frac{\dot{l}_2}{l_1 l_2^2}(\frac{\xi_1^2}{2} + W_2(\cdot)). \quad (19)$$

The inequality above is derived by neglecting the negative terms that are related to $\dot{l}_1$ and using the facts that $-k_1^2 W_2(\cdot) \leq -W_2(\cdot)$ and $\frac{1}{l_1} - \frac{1}{l_1(t-d)} \leq 0$ (see (22)). From (19), it is not difficult to show that the dynamic state compensator

$$\dot{l}_1 = \max\{-l_1^2 + l_1\rho_1(k_1,x_1),\ 0\},\ l_1(0) = 1, \quad (20)$$
$$\rho_1(k_1,x_1) = 2\Big[\Upsilon_{22}(\cdot) + \Upsilon_{22}^*(\cdot) + \Phi_{22}(\cdot) + \Phi_2^*(\cdot)\Big] \quad (21)$$

can counteract the effect of the time-delay nonlinearity. In fact, by construction the gain $l_1$ satisfies

$$0 \leq \dot{l}_1 \leq l_1\rho_1(\cdot),\ \dot{l}_1 \geq -l_1^2 + l_1\rho_1(\cdot), l_1 \geq l_1(t-d) \geq 1 \quad (22)$$

As a consequence,

$$-\frac{\dot{l}_1}{2l_1^2}\xi_1^2 \leq \xi_1^2 - \frac{1}{2l_1}\xi_1^2\rho_1(k_1,x_1) \quad (23)$$

Moreover,

$$\frac{2\bar{c}_2}{l_1}k_1^2\left|\xi_2^{2-1/p_1}(x_3^{p_2} - x_3^{*p_2})\right| \leq \bar{c}_2 k_1^2\xi_2^2 + c_3 k_1^2\xi_3^2, \quad (24)$$

where $\xi_3 = x_3^{p_1p_2} - x_3^{*p_1p_2}$, $\bar{c}_2$ and $c_3$ are positive constants.

Substituting (23) and (24) into (19), we arrive at

$$\dot{V}_{2LK} \leq -n\xi_1^2 - (n-1)k_1^2\xi_2^2 + (\theta_1 N(k_1)+1)\dot{k}_1 + c_3k_1^2\xi_3^2$$
$$+\frac{k_1^2}{l_1}(1+\frac{1}{l_2})\theta_2\xi_2^{2-1/p_1}x_3^* p_2 + k_1^2\xi_2^2\Big[c_2 + \bar{c}_2 + n - 1$$
$$+\Upsilon_{21}(\cdot) + \Upsilon_{21}^*(\cdot) + \Phi_{21}(\cdot)\Big] - \frac{\dot{l}_2}{l_1 l_2^2}(\frac{\xi_1^2}{2} + W_2(\cdot)) \quad (25)$$

Similar to Step 1, because of the unknown sign of $\theta_2$, we design the virtual controller

$$x_3^{*p_2} = l_1 N(k_2)\xi_2^{1/p_1}[\bar{c}_2 + n + \Upsilon_{21}(\cdot) + \Upsilon_{21}^*(\cdot) + \Phi_{21}(\cdot)]$$
$$:= l_1 N(k_2)(\xi_2\beta_2(k_1,x_1,x_2))^{1/p_1}$$
$$\dot{k}_2 = (1+\frac{1}{l_2})\xi_2^2\beta_2^{1/p_1}(k_1,x_1,x_2), \quad k_2(0) = 1. \quad (26)$$

with the Nussbaum gain $k_2$ that is updated dynamically. Clearly, the dynamic compensator (26) leads to

$$\dot{V}_{2LK} \leq -(n-1)(\xi_1^2 + k_1^2\xi_2^2) + (\theta_1 N(k_1)+1)\dot{k}_1$$
$$+c_3k_1^2\xi_3^2 + (\theta_2 N(k_2)+1)k_1^2\dot{k}_2 - \frac{\dot{l}_2}{l_1 l_2^2}(\frac{\xi_1^2}{2} + W_2(\cdot)). \quad (27)$$

**Inductive Step:** At step $i-1$, assume that there are a Lyapunov-Krasovskii functional $V_{(i-1)LK}$, a set of dynamic gains $l_j(\cdot) \geq 1$, $j = 1,\cdots,i-1$, updated by

$$\begin{aligned}
\dot{l}_1 &= \max\{-l_1^2 + l_1\rho_1(k_1,x_1), 0\},\\
\dot{l}_2 &= \max\{-\alpha_2 l_2^2 + l_2\rho_2(l_1,k_1,k_2,x_1,x_2), 0\},\\
&\ \vdots\\
\dot{l}_{i-2} &= \max\{-\alpha_{i-2}l_{i-2}^2 + l_{i-2}\rho_{i-2}(\bar{l}_{i-3},\bar{k}_{i-2}\bar{x}_{i-2}), 0\},
\end{aligned} \quad (28)$$

with $\alpha_j = 1/(2^{p_1\cdots p_{j-1}} - 1)$, and a set of non-smooth but $C^0$ virtual controllers $x_1^*,\cdots,x_i^*$, with the Nussbaum gains, given by

$$\begin{array}{lll}
x_1^* & = 0 & \xi_1 = x_1 - x_1^* \\
x_2^{*p_1} & = \xi_1 N(k_1)\beta_1(x_1) & \xi_2 = x_2^{p_1} - x_2^{*p_1} \\
\dot{k}_1 & = (1+\frac{1}{l_1})\xi_1^2\beta_1(\cdot) & \\
\vdots & & \vdots \\
x_i^{*p_1\cdots p_{i-1}} & = (l_1\cdots l_{i-2}N(k_{i-1}))^{p_1\cdots p_{i-2}} & \xi_i = x_i^{p_1\cdots p_{i-1}} \\
& \cdot\xi_{i-1}\beta_{i-1}(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1}) & -x_i^{*p_1\cdots p_{i-1}} \\
\dot{k}_{i-1} & = (1+\frac{1}{l_{i-1}})\xi_{i-1}^2\beta_{i-1}^{1/p_1\cdots p_{i-2}}(\cdot) &
\end{array} \quad (29)$$

with $\rho_j(\cdot) > 0$ and $\beta_j(\cdot) > 0$ being smooth functions, such that

$$\dot{V}_{(i-1)LK} \leq -(n-i+2)\sum_{j=1}^{i-1}\Big[\prod_{m=0}^{j-1}k_m^2\xi_j^2\Big] + c_i k_1^2\cdots k_{i-2}^2\xi_i^2 \quad (30)$$
$$+\sum_{j=1}^{i-1}\Big[(\theta_j N(k_j)+1)\prod_{m=0}^{j-1}k_m^2\dot{k}_j\Big] - \frac{\dot{l}_{i-1}(\frac{\xi_1^2}{2} + \Sigma_{j=2}^{i-1}W_j(\bar{l}_{j-2},\bar{k}_{j-1},\bar{x}_j))}{l_1\cdots l_{i-2}l_{i-1}^2}$$

where $c_i > 0$ is a constant and $k_0 = 1$. Clearly, (30) reduces to (27) when $i = 3$.

We claim that (30) also holds at Step $i$. To prove this claim, consider the Lyapunov-Krasovskii functional

$$V_i = V_{(i-1)LK} + \frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}W_i(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i) + \frac{1}{l_1\cdots l_i}\Big[\frac{\xi_1^2}{2}$$
$$+\Sigma_{j=2}^{i-1}W_j(\bar{l}_{j-2},\bar{k}_{j-1},\bar{x}_j) + k_1^2\cdots k_{i-1}^2 W_i(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i)\Big], \quad (31)$$
$$W_i = \int_{x_i^*}^{x_i}\left(s^{p_1\cdots p_{i-1}} - x_i^{*p_1\cdots p_{i-1}}\right)^{2-1/(p_1\cdots p_{i-1})}ds,$$

where $l_i(\cdot) \geq 1$ is a dynamic gain to be designed.

Similar to the argument in Step 2, one can show that $W_i(\cdot) = W_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i)$ is $C^1$. Moreover,

$$\begin{aligned}
\frac{\partial W_i}{\partial x_i} &= \xi_i^{2-1/(p_1\cdots p_{i-1})} \\
\frac{\partial W_i}{\partial x_j} &= -\left(2 - \frac{1}{p_1\cdots p_{i-1}}\right)\frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial x_j} U_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i) \\
\frac{\partial W_i}{\partial k_j} &= -\left(2 - \frac{1}{p_1\cdots p_{i-1}}\right)\frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial k_j} U_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i) \\
\frac{\partial W_i}{\partial l_j} &= -\left(2 - \frac{1}{p_1\cdots p_{i-1}}\right)\frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial l_j} U_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i) \\
& m_i(x_i - x_i^*)^{2p_1\cdots p_{i-1}} \le W_i(\cdot) \le (2^{p_1\cdots p_{i-1}} - 1)\xi_i^2
\end{aligned} \tag{32}$$

where $U_i = \int_{x_i^*}^{x_i}\left(s^{p_1\cdots p_{i-1}} - x_i^{*p_1\cdots p_{i-1}}\right)^{1-\frac{1}{(p_1\cdots p_{i-1})}} ds$ for $1 \le j \le i-1$ and a positive constant $m_i$.

Analogous to the derivation of (19), using the facts that $l_j \ge 1$ and $-k_1^2\cdots k_{i-1}^2 W_i(\cdot) \le -W_i(\cdot)$, we deduce from (30)-(32) that (by neglecting the negative terms which are related to $\dot{l}_j, j = 1, \ldots, i-1$)

$$\begin{aligned}
\dot{V}_i \le& -(n-i+2)\Sigma_{j=1}^{i-1}\Big[\prod_{m=0}^{j-1} k_m^2\xi_j^2\Big] + \Sigma_{j=1}^{i-1}\Big[(\theta_j N(k_j)+1)\prod_{m=0}^{j-1} k_m^2 \dot{k}_j\Big] \\
&+ c_i k_1^2\cdots k_{i-2}^2\xi_i^2 - \frac{\dot{l}_{i-1}}{l_1\cdots l_{i-2}l_{i-1}^2}\left[\frac{\xi_1^2}{2} + \Sigma_{j=2}^{i-1}W_j(\bar{l}_{j-2}, \bar{k}_{j-1}, \bar{x}_j)\right] \\
&+ \frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}\left(1 + \frac{1}{l_i}\right)\Bigg[\left|\xi_i^{2-\frac{1}{(p_1\cdots p_{i-1})}} f_i(\cdot)\right| + \theta_i\xi_i^{2-\frac{1}{(p_1\cdots p_{i-1})}}(x_{i+1}^{*p_i} \\
&- x_{i+1}^{p_i} + x_{i+1}^{*p_i}) + \left|\Sigma_{j=1}^{i-1}\frac{\partial W_i}{\partial x_j}\dot{x}_j + \Sigma_{j=1}^{i-1}\frac{\partial W_i}{\partial k_j}\dot{k}_j + \Sigma_{j=1}^{i-2}\frac{\partial W_i}{\partial l_j}\dot{l}_j\right|\Bigg] \\
&+ \frac{2}{l_1\cdots l_{i-1}}\left(\Sigma_{j=1}^{i-1}(k_j\dot{k}_j\prod_{\substack{m=1\\m\ne j}}^{i-1} k_m^2)\right)W_i(\cdot) + \frac{1}{l_1\cdots l_i} \\
&\cdot\left[\Sigma_{j=2}^{i-1}\left(\Sigma_{m=1}^{j}\frac{\partial W_j}{\partial x_m}\dot{x}_m + \Sigma_{m=1}^{j-1}\frac{\partial W_j}{\partial k_m}\dot{k}_m + \Sigma_{m=1}^{j-2}\frac{\partial W_j}{\partial l_m}\dot{l}_m\right) + \xi_1\dot{x}_1\right] \\
&- \frac{\dot{l}_i}{l_1\cdots l_{i-1}l_i^2}\left(\frac{\xi_1^2}{2} + \Sigma_{j=1}^{i}W_j(\cdot)\right).
\end{aligned} \tag{33}$$

Using an argument similar to Zhang, Lin and Lin (2017), we obtain the estimations (34)-(38) (see the appendix for details):

$$\begin{aligned}
&\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}\left|\xi_i^{2-1/(p_1\cdots p_{i-1})} f_i\right| \le k_1^2\cdots k_{i-1}^2\xi_i^2 \\
&\cdot\Upsilon_{i1}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i) + \xi_i^2(t-d) \\
&\cdot\Upsilon_{i1}^*(\bar{l}_{i-2}(t-d), \bar{k}_{i-1}(t-d), \bar{x}_i(t-d)) + \frac{1}{l_1\cdots l_{i-1}} \\
&\cdot\left[\xi_1^2 + \Sigma_{j=2}^{i-1}\left(x_j - x_j^*\right)^{2p_1\cdots p_{j-1}}\right]\Upsilon_{i2}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) \\
&+ \frac{1}{l_1\cdots l_{i-1}}\left[\Sigma_{j=2}^{i-1}\left(x_j(t-d) - x_j^*(t-d)\right)^{2p_1\cdots p_{j-1}}\right. \\
&\left.+ \xi_1^2(t-d)\right]\Upsilon_{i2}^*(\bar{l}_{i-2}(t-d), \bar{k}_{i-1}(t-d), \bar{x}_{i-1}(t-d)),
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}\left|\Sigma_{j=1}^{i-1}\frac{\partial W_i}{\partial x_j}\dot{x}_j + \Sigma_{j=1}^{i-1}\frac{\partial W_i}{\partial k_j}\dot{k}_j + \Sigma_{j=1}^{i-2}\frac{\partial W_i}{\partial l_j}\dot{l}_j\right| \\
&\le k_1^2\cdots k_{i-1}^2\xi_i^2\Phi_{i1}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i) + \frac{1}{l_1\cdots l_{i-1}}\left[\xi_1^2\right. \\
&\left.+ \Sigma_{j=2}^{i-1}\left(x_j - x_j^*\right)^{2p_1\cdots p_{j-1}}\right]\Phi_{i2}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) \\
&+ \frac{1}{l_1\cdots l_{i-1}}\left[\Sigma_{j=2}^{i-1}\left(x_j(t-d) - x_j^*(t-d)\right)^{2p_1\cdots p_{j-1}}\right. \\
&\left.+ \xi_1^2(t-d)\right]\Phi_{i2}^*(\bar{l}_{i-2}(t-d), \bar{k}_{i-1}(t-d), \bar{x}_{i-1}(t-d)),
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\frac{2}{l_1\cdots l_{i-1}}\left[\Sigma_{j=1}^{i-1}\left(2k_j\dot{k}_j\prod_{\substack{m=1\\m\ne j}}^{i-1} k_m^2\right)\right]W_i(\cdot) \\
&\qquad \le k_1^2\cdots k_{i-1}^2\xi_i^2\omega_i(\bar{l}_{i-3}, \bar{k}_{i-2}, \bar{x}_{i-1}),
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}|\theta_i\xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}}(x_{i+1}^{p_i} - x_{i+1}^{*p_i})| \\
&\qquad \le k_1^2\cdots k_{i-1}^2(\bar{c}_i\xi_i^2 + .c_{i+1}\xi_{i+1}^2),
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\frac{1}{l_1\cdots l_i}\left|\xi_1\dot{x}_1 + \Sigma_{j=2}^{i-1}\Big[\Sigma_{m=1}^{j}\frac{\partial W_j}{\partial x_m}\dot{x}_m + \Sigma_{m=1}^{j-1}\frac{\partial W_j}{\partial k_m}\dot{k}_m\right. \\
&\left.+ \Sigma_{m=1}^{j-2}\frac{\partial W_j}{\partial l_m}\dot{l}_m\Big]\right| \le \frac{1}{l_1\cdots l_{i-1}}\Psi_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) \\
&\cdot\left[\xi_1^2 + \Sigma_{j=2}^{i-1}\left(x_j - x_j^*\right)^{2p_1\cdots p_{j-1}}\right] + k_1^2\cdots k_{i-1}^2\xi_i^2 \\
&+ \frac{1}{l_1\cdots l_{i-1}}\Psi_i^*(\bar{l}_{i-2}(t-d), \bar{k}_{i-1}(t-d), \bar{x}_{i-1}(t-d)) \\
&\cdot\left[\xi_1^2(t-d) + \Sigma_{j=2}^{i-1}\left(x_j(t-d) - x_j^*(t-d)\right)^{2p_1\cdots p_{j-1}}\right]
\end{aligned} \tag{38}$$

where $\Upsilon_{ij}(\cdot) \ge 0$, $\Upsilon_{ij}^*(\cdot) \ge 0$, $\Phi_{ij}(\cdot) \ge 0$, $\Phi_{ij}^*(\cdot) \ge 0$, $\Psi_i(\cdot) \ge 0$ and $\Psi_i^*(\cdot) \ge 0, \omega_i(\cdot) \ge 0$ $j = 1, 2$, are smooth functions.

With the help of the bounds $\Upsilon_{ij}^*(\cdot)$, $\Phi_{ij}^*(\cdot)$ and $\Psi_i^*(\cdot)$ thus obtained, which are related to the delay terms, we construct the Lyapunov-Krasovskii functional

$$\begin{aligned}
V_{iLK} =& V_i + \int_{t-d}^{t}\xi_i^2(s)\Upsilon_{i1}^*(\bar{l}_{i-2}(s), \bar{k}_{i-1}(s), \bar{x}_i(s))ds + \\
&\int_{t-d}^{t}\frac{1}{l_1(s)\cdots l_{i-1}(s)}\left[\xi_1^2(s) + \Sigma_{j=2}^{i-1}\left(x_j(s) - x_j^*(s)\right)^{2p_1\cdots p_{j-1}}\right] \\
&\cdot\Big[\Upsilon_{i2}^*(\bar{l}_{i-2}(s), \bar{k}_{i-1}(s), \bar{x}_{i-1}(s)) + \Phi_{i2}^*(\bar{l}_{i-2}(s), \bar{k}_{i-1}(s), \bar{x}_{i-1}(s)) \\
&+ \Psi_i^*(\bar{l}_{i-2}(s), \bar{k}_{i-1}(s), \bar{x}_{i-1}(s))\Big]ds
\end{aligned} \tag{39}$$

From (33)-(38) and the fact that $\frac{1}{l_1\cdots l_{i-1}(t)} \le \frac{1}{l_1(t-d)\cdots l_{i-1}(t-d)}$ and $k_i \ge 1, i = 1, \ldots, i-1$, A straightforward but tedious calculation gives

$$\begin{aligned}
\dot{V}_{iLK} \le& -(n-i+2)\Sigma_{j=1}^{i-1}\Big[\prod_{m=0}^{j-1} k_m^2\xi_j^2\Big] + \Sigma_{j=1}^{i-1}\Big[(\theta_j N(k_j)+1)\prod_{m=0}^{j-1} k_m^2\dot{k}_j\Big] \\
&- \frac{\dot{l}_{i-1}(\frac{\xi_1^2}{2} + \Sigma_{j=2}^{i-1}W_j(\cdot))}{l_1\cdots l_{i-2}l_{i-1}^2} + \frac{\left[\xi_1^2 + \Sigma_{j=2}^{i-1}\left(x_j - x_j^*\right)^{2p_1\cdots p_{j-1}}\right]}{l_1\cdots l_{i-1}} \\
&\cdot\Big[\Upsilon_{i2}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) + \Upsilon_{i2}^*(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) + \Phi_{i2}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) \\
&+ \Phi_{i2}^*(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) + \Psi_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) + \Psi_i^*(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1})\Big] \\
&+ \frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}\left(1 + \frac{1}{l_i}\right)\theta_i\xi_i^{2-1/(p_1\cdots p_{i-1})}x_{i+1}^{*p_i} + c_{i+1}k_1^2\cdots k_{i-1}^2\xi_{i+1}^2 \\
&+ k_1^2\cdots k_{i-1}^2\xi_i^2\Big[1 + c_i + \bar{c}_i + \Upsilon_{i1}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i) + \Upsilon_{i1}^*(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i) \\
&+ \Phi_{i1}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i) + \omega_i(\bar{l}_{i-3}, \bar{k}_{i-2}, \bar{x}_{i-1})\Big] - \frac{\dot{l}_i[\frac{\xi_1^2}{2} + \Sigma_{j=2}^{i}W_j(\cdot)]}{l_1\cdots l_{i-1}l_i^2}
\end{aligned} \tag{40}$$

Based on (40), one can design the delay-free gain update law

$$\begin{aligned}
\dot{l}_{i-1} &= \max\{-\alpha_{i-1}l_{i-1}^2 + l_{i-1}\rho_{i-1}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}), 0\}, \ l_{i-1}(0) = 1 \\
\rho_{i-1}(\cdot) &= \frac{1}{M_{i-1}}\left[\Upsilon_{i2}(\cdot) + \Upsilon_{i2}^*(\cdot) + \Phi_{i2}(\cdot) + \Phi_{i2}^*(\cdot) + \Psi_i(\cdot) + \Psi_i^*(\cdot)\right],
\end{aligned} \tag{41}$$

where $\alpha_{i-1} = \frac{1}{(2^{p_1\cdots p_{i-2}} - 1)}$ and $M_{i-1} = \min\{\frac{1}{2}, m_2, \cdots, m_{i-1}\}$

By construction, the gain thus constructed satisfies

$$0 \le \dot{l}_{i-1} \le l_{i-1}\rho_{i-1}(\cdot), \quad \dot{l}_{i-1} \ge -\alpha_{i-1}l_{i-1}^2 + l_{i-1}\rho_{i-1}(\cdot) \tag{42}$$

Using (32) and (42), it is not difficult to prove that

$$\frac{-\dot{l}_{i-1}}{l_1\cdots l_{i-2}l_{i-1}^2}(\frac{\xi_1^2}{2}+\Sigma_{j=2}^{i-1}W_j(\cdot)) \le \Sigma_{j=1}^{i-1}[(\prod_{m=0}^{j-1}k_m^2)\xi_j^2]$$
$$-\frac{M_{i-1}\rho_{i-1}(\cdot)}{l_1\cdots l_{i-1}}\left[\xi_1^2+\Sigma_{j=2}^{i-1}\left(x_j-x_j^*\right)^{2p_1\cdots p_{j-1}}\right] \tag{43}$$

Substituting (41) into (40) yields

$$\dot{V}_{iLK} \le -[n-i+1]\Sigma_{j=1}^{i}\left[\prod_{m=0}^{j-1}k_m^2\xi_j^2\right]+\Sigma_{j=1}^{i-1}[(\theta_j N(k_j)+1)\prod_{m=0}^{j-1}k_m^2\dot{k}_j]$$
$$+\frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}(1+\frac{1}{l_i})\xi_i^{2-1/(p_1\cdots p_{i-1})}x_{i+1}^{*p_i}+k_1^2\cdots k_{i-1}^2\xi_i^2$$
$$\cdot\left[2+c_i+\bar{c}_i+n-i+\Upsilon_{i1}(\cdot)+\Upsilon_{i1}^*(\cdot)+\Phi_{i1}(\cdot)+\omega_i(\cdot)\right]$$
$$-\frac{\dot{l}_i}{l_1\cdots l_{i-1}l_i^2}(\frac{\xi_1^2}{2}+\Sigma_{j=2}^{i}W_j(\cdot))+c_{i+1}k_1^2\cdots k_{i-1}^2\xi_{i+1}^2. \tag{44}$$

In view of (44), one can design the non-smooth virtual controller with the Nussbaum gain

$$x_{i+1}^{*p_i} = l_1\cdots l_{i-1}N(k_i)\xi_i^{1/(p_1\cdots p_{i-1})}\Big[2+c_i+\bar{c}_i+n-i$$
$$+\Upsilon_{i1}(\cdot)+\Upsilon_{i1}^*(\cdot)+\Phi_{i1}(\cdot)+\omega_i(\cdot)\Big]$$
$$:= l_1\cdots l_{i-1}N(k_i)(\xi_i\beta_i(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i))^{1/(p_1\cdots p_{i-1})}$$
$$\dot{k}_i = (1+\frac{1}{l_i})\xi_i^2\beta_i(\cdot)^{1/(p_1\cdots p_{i-1})} \tag{45}$$

Substituting (45) into (44), we can show that (30) holds at Step $i$.

Using the claim for $i=n+1$ with $u=x_{n+1}=x_{n+1}^*$, we conclude that the dynamic state feedback controller that is composed of (28) with $i=n+1$ and

$$u = (l_1\cdots l_{n-1}N(k_n))^{\frac{1}{p_n}}\left(\xi_n\beta_n(\bar{l}_{n-2},\bar{k}_{n-1},x)\right)^{\frac{1}{(p_1\cdots p_n)}}$$
$$\dot{k}_n = \xi_n^2\beta_n(\bar{l}_{n-2},\bar{k}_{n-1},x)^{\frac{1}{(p_1\cdots p_{n-1})}} \tag{46}$$

is such that

$$\dot{V}_{nLK} \le -\Sigma_{j=1}^{n}[(\prod_{m=0}^{j-1}k_m^2)\xi_j^2]+\Sigma_{j=1}^{n}[(\theta_j N(k_j)+1)(\prod_{m=0}^{j-1}k_m^2)\dot{k}_j]. \tag{47}$$

## 4. STATE REGULATION WITH BOUNDEDNESS

We now use the inequality (47) to complete the proof of Theorem 6. In particular, it is shown that the proposed dynamic state feedback controller (46) and (28) can regulate the system state to the origin, while maintaining the boundedness of the closed-loop system.

First of all, we can establish, based on the Lyapunov inequalities (30) and (47), the following lemma.

*Lemma 4.1.* The Nussbaum gains $k_i(t), i=1\cdots n$, given by (45) are bounded $\forall t\in[0,+\infty)$.

The proof of Lemma 4.1 requires delicate and tedious analyses and can be carried out in a fashion similar to the one in the appendix of Pongvuthithum, Rattanamongkhonkun and Lin (2018). The details are omitted due to the limited space.

With the aid of the boundedness of $k_i(t), 1\le i\le n$, we deduce from (45) that $\xi_i^2(t)\le \dot{k}_i(t)$ because, by construction, $\beta_i(\cdot)\ge 1$ and $l_i(t)\ge 1$. Hence, $\int_0^{+\infty}\xi_i^2 ds \le k_i(+\infty)-k_i(0)=c$. On the other hand, (47) and the boundedness of $k_i(t), 1\le i\le n$, imply that

$$V_{nLK}(t) \le \Sigma_{j=1}^{n}\int_0^t|\theta_j N(k_j(s))+1|(\prod_{m=0}^{j-1}k_m^2(s))\dot{k}_j(s)ds+V_{nLK}(0)$$
$$\le c_1\Sigma_{j=1}^{n}\int_0^t\dot{k}_j(s)ds+c_2\le C. \tag{48}$$

In view of (39) and (31), it is clear that the boundedness of $V_{nLK}(\cdot)$ on $[0,+\infty)$ implies the boundedness of $x_1$, $\frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}W_i(\cdot)$, $i=2,\cdots,n$. Using the estimation of $W_i(\cdot)$ in (32), one concludes that $x_1$ and $\frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}(x_i-x_i^*)^{2p_i\cdots p_i}$, $i=2,\cdots,n$, are also bounded. Because $x_1$ and $k_1$ are bounded and the gain $l_1(\cdot)$ given by (52)-(21) is monotone non-deceasing, then $l_1(\cdot)$ must be bounded. If not, $\lim_{t\to+\infty}l_1(t)=+\infty$. By continuity of $\rho_1(\cdot)$, $\rho_1(k_1,x_1)$ is bounded. Consequently, there is a time instant $T>0$ such that $-l_1^2+l_1\rho_1(k_1,x_1)\le 0$ on $[T,+\infty)$. This, together with (22), yields $\dot{l}_1=0$ on $[T,+\infty)$, which contradicts to the unboundedness of $l_1(\cdot)$. In conclusion, $l_1(\cdot)$ is bounded. The boundedness of $l_1(\cdot)$ and $k_1$ implies the boundedness of $x_2^*$ as well as $x_2-x_2^*$. As a such, $x_2$ is also bounded. Similarly, one can prove the boundedness of $l_i(\cdot)$ and $x_i$ in the following recursive manner: $x_2\to l_2\to x_3\to\cdots\to l_{n-1}\to x_n$, by the boundedness of $k_i(\cdot), i=1,\cdots,n$, (28) and the estimation (32). Therefore, all the signals of the closed-loop system (1)-(46)-(28) are bounded $\forall t\in[0,+\infty)$.

To prove the convergence of the system state, we observe that $\dot{\xi}_i$, $i=1,\cdots,n$ are also bounded and $\int_0^{+\infty}\xi_i^2(t)dt<+\infty$. By the Barbalat's lemma, it is concluded that $\xi_i$, $i=1,\cdots,n$ converge to zero. This, in view of the coordinate transformation (29), implies that all the states $x_1(t),\cdots,x_n(t)$ converge to zero as well, thus completing the proof of Theorem 3.1. □

Because the proposed nonsmooth control scheme is based on the Lyapunov-Krasovskii functional method, it is not surprising that Theorem 6 is robust with respect to the uncertainty. With this observation in mind, Theorem 6 can be extended to a larger family of uncertain time-delay systems dominated by a homogeneous system with time-delay. In fact, the following more general result also holds.

*Theorem 7.* Consider a family of uncertain time-delay systems with unknown control directions:

$$\dot{x}_i = \theta_i x_{i+1}^{p_i}+\phi_i(x,x(t-d),t), \quad i=1,\cdots,n, \tag{49}$$

where $x_{n+1}=u$ and $\phi_i:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$, is a continuous mapping. Assume that the uncertain function $\phi_i$, $i=1,\cdots,n$, satisfies the homogeneous growth condition

$$|\phi_i(\cdot)|\le\gamma_i(\bar{x}_i,\bar{x}_i(t-d))\Big(|x_1|^{\frac{1}{p_1\cdots p_{i-1}}}+|x_2|^{\frac{1}{p_2\cdots p_{i-1}}}+\cdots+|x_{i-1}|^{\frac{1}{p_{i-1}}}$$
$$+|x_i|+|x_1(t-d)|^{\frac{1}{p_1\cdots p_{i-1}}}+\cdots+|x_{i-1}(t-d)|^{\frac{1}{p_{i-1}}}+|x_i(t-d)|\Big) \tag{50}$$

with $\gamma_i(\bar{x}_i,\bar{x}_i(t-d))\ge 0$ being a known smooth function. Then, there is a delay-free, nonsmooth but $C^0$ dynamic state feedback (8) that steers the state $x$ to zero and keeps the boundedness of the closed-loop system (8)-(49). □

Under the homogeneous growth condition (50), the proof of Theorem 7 can be carried out, with some subtle modifications, by means of an argument analogue to that of Theorem 6. For this reason, the details are left to the reader as an exercise.

*Remark 8.* When the nonlinear system (1) or (49) has multiple delays, the design of a delay-independent controller remains almost same, except that multiple Lyapunov-Krasovskii functionals with different time-delays need to be used. Specifically, the Lyapunov-Krasovskii functional $\int_{t-d}^{t}K(s)ds$ should be replaced by $\int_{t-d_i}^{t}K(s)ds$ in the recursive design. Of course, a similar philosophy can be employed to handle the general case when every subsystem of (1) involves different time delays.

*Remark 9.* The assumption that the bound $C$ of unknown coefficients $\theta_i,\ 1 \leq i \leq n$ is known is used for a technical convenience and can indeed be removed. When the bound $C$ is unknown, a similar design procedure can be carried out with slightly different estimations of the right hand side of $\dot{V}_{iLK}$ in (44) so that the term $(\theta_j N(k_j)+1)$ is replaced by $(\theta_j N(k_j)+C_j)$, where $C_j$ is an unknown constant. Due to the characteristics of the Nussbaum function and the monotone property of the adaptive gains $k_j,\ 1 \leq j \leq n$, the same argument in Appendix B can also be used for the stability proof.

Finally, we present a simple but nontrivial example that demonstrates how Nussbaum gains need to be introduced to handle the problem of unknown control directions.

*Example 10.* Consider a time-delay system in the plane, with strong nonlinearity and unknown control directions, of the form

$$\dot{x}_1 = \theta_1 x_2^3 + x_1, \quad \dot{x}_2 = \theta_2 u + \frac{1}{2}x_2^3(t-d), \tag{51}$$

where $\theta_1, \theta_2 \neq 0$ are unknown constants whose signs are also unknown (either positive or negative), and represents unknown directions of the actuator. Note that the time-delay system under consideration involves not only an unknown control direction but also strong nonlinearities. The latter requires the use of a nonsmooth rather than smooth feedback control strategy. In fact, even in the case when control directions are known (e.g., $\theta_1 = \theta_2 = 1$) and no time delay is involved (i.e., $d = 0$), it is known that the planar system cannot be controlled by any smooth state feedback, even locally, and a nonsmooth feedback must be employed.

Following the control scheme proposed in section 3, we first consider the Lyapunov function $V_1(x_1, l_1) = \frac{1}{2}(1+\frac{1}{l_1})\xi_1^2$, where $\xi_1 = x_1$ and the gain $l_1$ is updated by

$$\dot{l}_1 = \max\{-l_1^2 + l_1\rho(k_1, x_1),\ 0\}, \quad l_1(0) = 1, \tag{52}$$

with $\rho_1(\cdot) \geq 0$ being a smooth function to be determined later on.

For the $x_1$-subsystem, it is clear that the nonsmooth virtual control law $x_2^{*3} = 2x_1 N(k_1)$, with $\dot{k}_1 = 2(1+\frac{1}{l_1})x_1^2$, globally asymptotically regulates it.

Define $\xi_2 = x_2^3 - x_2^{*3} = x_2^3 - 2x_1 N(k_1)$. From (52), it is easy to see that $l_1(\cdot) \geq 1$ and $\dot{l}_1 \geq -l_1^2 + l_1\rho_1(k_1, x_1)$. Moreover,

$$\dot{V}_1 \leq -2x_1^2 + (\theta_1 N(k_1)+1)\dot{k}_1 - \frac{1}{2l_1}\xi_1^2\rho_1(k_1, x_1). \tag{53}$$

Then, consider the Lyapunov-Krasovskii functional

$$V_{2LK} = V_1(\cdot) + \frac{k_1^2}{l_1}\int_{x_2^*}^{x_2}(s^3 - x_2^{*3})^{2-\frac{1}{3}}ds + \int_{t-d}^{t}\frac{\left(\xi_2^6(s)+2(k_1^6x_1^3(s))^2\right)}{l_1(s)}ds \tag{54}$$

Following the design procedure in Step 2, one can find a dynamic state compensator that consists of (52) and

$$\begin{aligned} u &= N(k_2)\xi_2^{1/3}\left(2(k_1^2+2k_1)^2 + \frac{10}{3}(1+\frac{1}{l_1})x_1^2 + l_1 + \xi_2^4\right) \\ \dot{k}_2 &= \frac{1}{l_1}\xi_2^2\left(2(k_1^2+2k_1)^2 + \frac{10}{3}(1+\frac{1}{l_1})x_1^2 + l_1 + \xi_2^4\right) \end{aligned} \tag{55}$$

with $\rho_1(k_1, x_1) = 2(2x_1^4k_1^{12} + 4k_1^6 + \frac{5}{3}(1+\frac{1}{l_1})x_1^2k_1^2)$ in (52) and $N(k_2) = k_2^2\cos(k_2)$, such that

$$\dot{V}_{2LK} \leq -x_1^2 - k_1^2\xi_2^2 + (\theta_1 N(k_1)+1)\dot{k}_1 + \frac{(\theta_2 N(k_2)+1)k_1^2\dot{k}_2}{l_1},$$

from which it is deduced, as shown in Section 4, that the delay-free controller (55) and (52) achieves asymptotic state regulation and maintains the boundedness of the closed-loop system (51), (52) and (55), without the information of the sign of the parameter $\theta_i,\ i = 1, 2$.

## 5. CONCLUDING REMARKS

In this paper, we have presented a delay-free, non-smooth dynamic state feedback scheme to control a family of uncertain time-delay systems with strong nonlinearities and unknown control directions. To cope with the effects of time-delay nonlinearities and unknown control directions, we have introduced, respectively, two sets of gains that need to be updated online, in a dynamic manner. One of them is the Nussbaum-type gains from universal control Nussbaum (1983), making it possible to mitigate the effect of unknown control directions, while the other one is borrowed the idea from the dynamic state feedback control method Zhang, Lin and Lin (2017), which is able to counteract the time-delay effects via a delay-free nonsmooth controller. Global asymptotic state regulation with boundedness of the closed-loop system has been proved to be possible, thanks to the construction of a set of new Lyapunov-Krasovskii functionals that are different from the previous ones in the literature, due to the involvement of the Nussbaum gains.

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