String and robust stability of connected vehicle systems with delayed feedback

Gopal Krishna Kamath, Krishna Jagannathan and Gaurav Raina

Department of Electrical Engineering,
Indian Institute of Technology Madras, Chennai 600 036, India.
Email: {ee12d033, krishnaj, gaurav}@ee.iitm.ac.in

Abstract: Two important characteristics of a connected vehicle system are string stability (*i.e.,* disturbance in state variables should not propagate along the platoon) and robustness to parametric uncertainties. In this paper, we study these properties for a platoon of vehicles driving on a single-lane straight road. Specifically, we model individual vehicle's dynamics by the Classical Car-Following Model (CCFM) and the Modified Optimal Velocity Model (MOVM). These models capture the reaction delays inherent in such a setting. First, we focus on string stability. For both models, we derive a sufficient condition for string stability. We then concentrate on the robustness to uncertainties in the parameters of the said models. For both models, we derive bounds on the reaction delay to ensure (a) pairwise robust stability, and (b) platoon robust stability, when the remaining model parameters lie in an interval. Finally, we compare our results with conditions for these models to be locally asymptotically stable. This brings forth the additional constraints imposed on the reaction delay to achieve string and robust stability. Our results may provide design guidelines for futuristic connected vehicle systems. Additionally, from a technological perspective, our results suggest the sufficiency of using only on-board sensors for all forms of local stability except platoon robust stability.

Keywords: Connected vehicles, Car-following models, Time delays, String stability, Robustness

1. INTRODUCTION

In the context of futuristic smart cities, intelligent transportation systems would play a central role. Specifically, the use of connected systems of autonomous vehicles is seen as a prospective solution to (a) reducing congestion via proper resource utilization, and (b) ensuring human safety (Greengard, 2015). Thus, gaining a good understanding of connected vehicle systems is imperative. To that end, such systems are popularly modeled as dynamical systems in the literature. Several models – known as *car-following models* – have been studied in the literature; see (Bando et al., 1998; Gazis et al., 1961; Kamath et al., 2015; Orosz and Stépán, 2006; Unwin and Duckstein, 1967; Zhang and Jarrett, 1997) and references therein.

An important characteristic of most car-following models is the spatially *local* interaction among vehicles. In particular, each vehicle updates its acceleration based on the acceleration, velocity and distance of the vehicle directly ahead. Hence, the primary focus in the literature has been on *pairwise stability* of car-following models; that is, does the interaction between consecutive vehicles lead to asymptotically stable dynamics for the given pair? However, it is known that a connected vehicle system which is pairwise stable need not be *string stable* (Peppard, 1974), *i.e.,* disturbances in the state variables may propagate down the platoon despite being pairwise stable. Further, it would also be useful to study robustness of connected vehicle systems to parametric uncertainties.

Motivated by the above stability considerations, we study string stability and robustness to parametric uncertainties of a platoon of vehicles traversing a single-lane straight road. We make use of two car-following models with delayed feedback to describe the vehicular dynamics – the Classical Car-Following Model (CCFM) (Kamath et al., 2016) and the Modified Optimal Velocity Model (MOVM) (Kamath et al., 2017). Our analyses may provide design guidelines for futuristic connected vehicle systems. Additionally, our results suggest that all forms of stabilities other than platoon robust stability can be achieved using only on-board sensors.

The CCFM, which was proposed in (Gazis et al., 1961), aims to mimic the behavior of a human driver. While early studies (see (Gazis et al., 1961; Herman et al., 1959; Unwin and Duckstein, 1967)) used transform techniques to analyze the CCFM, Zhang *et al.* modeled the CCFM as a dynamical system in addition to generalizing the model (Zhang and Jarrett, 1997). The CCFM was further generalized in (Kamath et al., 2015) and again in (Kamath et al., 2016), the last of which we study in this paper. On the other hand, the MOVM follows from a car-following model proposed in the Physics literature called the Optimal Velocity Model (OVM) (Bando et al., 1998). While the latter was proposed for vehicles traveling on a circular loop (thus yielding periodic boundary conditions), the former captures the dynamics of a vehicular platoon on a straight road.

For the CCFM and the MOVM, conditions for local stability, non-oscillatory convergence and the rate of convergence have been studied (Kamath et al., 2016, 2017). Therein, the authors also show that both models lose local stability via a Hopf bifurcation. Specifically, the impact of reaction delays on the qualitative dynamical properties of both models has been highlighted. Further, it is seen that integrating a Proportional Derivative Acceleration (PDA) controller in the OVM has a positive impact on its dynamical properties (Ge and Orosz, 2014). Motivated by this, Kamath *et al.* incorporate the PDA controller in the CCFM (Kamath et al., 2018). Therein, the authors analytically bring forth the degradation caused to the performance of the CCFM in several measures of interest, including string and robust stability.

In this paper, we first derive sufficient conditions for string stability of the CCFM and the MOVM. In the context of the CCFM, our results generalize those presented in (Sipahi and Niculescu, 2008; Zhang and Jarrett, 1997). In the context of the MOVM, to the best of our knowledge, our result is the first of its kind.

Next, we show that the CCFM cannot be robust stable for arbitrary non-negative values of the reaction delay. Hence, we find bounds on the reaction delay when other model parameters are uncertain in an interval. Specifically, we derive a sufficient condition each for the CCFM and the MOVM to be (a) pairwise, and (b) platoon robust stable.

Related work on string and robust stability

String stability has been studied both in the context of connected vehicles (Sipahi and Niculescu, 2008) and general interconnected systems (Feintuch and Francis, 2012; Swaroop and Hedrick, 1996). In the specific context of connected vehicles, Klinge *et al.* and Middleton *et al.* study models without delays (Klinge and Middleton, 2009; Middleton and Braslavsky, 2010) while Sipahi *et al.* account for reaction delays (Sipahi and Niculescu, 2008). Further, the authors in (Ploeg et al., 2014) have extended the classical definition of string stability based on the $\mathcal{H}_\infty$-norm to the general $\mathcal{L}_p$ string stability for non-linear systems. However, we restrict ourselves to the classical definition, and our work is closely related to that presented in (Sipahi and Niculescu, 2008). For a recent exposition of string stability as applied to connected vehicle systems, refer to (Besselink and Johansson, 2017) and references therein.

Robust stability analysis has been extensively studied for systems without delays; see (Mackenroth, 2013) and references therein. Several results have been extended to systems with delays in state and input; see (Kharitonov, 1999; Kharitonov and Melchor-Aguilar, 2000) and references therein. Additionally, the book chapter (Niculescu et al., 1998) serves as an excellent reference for robust stability analysis of time-delayed systems. We apply results from (Kharitonov, 1999; Kharitonov and Melchor-Aguilar, 2000) to the transportation scenario for obtaining bounds on the reaction delay, when other parameters lie in an interval.

The remainder of this paper is organized as follows. In Section 2, we describe the setting for our work, and describe the models to be analyzed. We then present our results on string and robust stability in Sections 3 and 4 respectively. We then compare various notions of stability in Section 5. Finally, we conclude the paper in Section 6.

2. MODELS

In this section, we explain the setting for our work and also describe the dynamical models for the vehicles.

2.1 Setting

We consider a platoon of connected vehicles traveling on a single-lane, straight road without overtaking. Specifically, we assume that N vehicles follow a lead vehicle, whose dynamics is assumed to be known. Further, we assume that each vehicle is well modeled by an ideal point, *i.e.,* it has zero length. We also assume an arbitrary reference on the road, and measure the distance of each vehicle with respect to this reference. In particular, for the i^{th} vehicle, we denote this distance at time t as $x_i(t)$. Following standard convention, we denote the velocity and acceleration of the i^{th} vehicle at time t by $\dot{x}_i(t)$ and $\ddot{x}_i(t)$ respectively. We also restrict ourselves to leader profiles such that, for some $0 < T < \infty$, $\ddot{x}_0(t) = 0$ and $\dot{x}_0(t) = \dot{x}_0 > 0\ \forall t \geq T$. That is, the lead vehicle reaches a steady cruising behavior in finite time.

Note that as $t \to \infty$, $x_i(t) \to \infty$ for each i. Hence, to work with bounded state variables, we define the relative spacing (headway) and the relative velocity between the $(i-1)^{th}$ and i^{th} vehicles at time t as $y_i(t) = x_{i-1}(t) - x_i(t)$ and $v_i(t) = \dot{y}_i(t) = \dot{x}_{i-1}(t) - \dot{x}_i(t)$ respectively. We work with these variables throughout the paper. We also use SI units throughout.

We now briefly describe the models that capture the dynamics of individual vehicles in the platoon. For a detailed discussion on the CCFM and the MOVM, the reader is referred to (Kamath et al., 2016) and (Kamath et al., 2017) respectively.

2.2 The Classical Car-Following Model (CCFM)

The Classical Car-Following Model (CCFM) was proposed by Gazis *et al.* (Gazis et al., 1961). A general version incorporating heterogeneity in reaction delays and headways, in addition to delayed self-velocity term was proposed in (Kamath et al., 2016). We make use of this general model, whose evolution equations are given by

$$\begin{aligned}\dot{v}_i(t) &= \beta_{i-1}(t-\tau_{i-1})v_{i-1}(t-\tau_{i-1}) - \beta_i(t-\tau_i)v_i(t-\tau_i),\\ \dot{y}_i(t) &= v_i(t),\end{aligned} \tag{1}$$

for $i \in \{1, 2, \ldots, N\}$. Here,

$$\beta_i(t) = \alpha_i \frac{(\dot{x}_0(t) - v_0(t) - \cdots - v_i(t))^m}{(y_i(t) + b_i)^l},$$

where, $\alpha_i > 0$ and $\tau_i \geq 0$ are the sensitivity coefficient and the reaction delay of the i^{th} vehicle respectively, and $l \geq 0$ and $m \in [-2, 2]$ are model non-linearity parameters. Further, $b_i > 0$ represents the desired headway between the $(i-1)^{th}$ and i^{th} vehicles. Note that, for the CCFM, $y_i(t)$ represents the variation of the headway about its equilibrium value b_i and does not itself represent the headway (Kamath et al., 2016).

The CCFM is described by a system of non-linear delay differential equations. We linearize (1) about a desired equilibrium and conduct a *local* analysis to obtain design guidelines. To that end, note that $v_i^* = 0$ and $y_i^* = 0$, for each i is an equilibrium for the CCFM. Linearizing system (1) about the said equilibrium, we obtain

$$\dot{v}_i(t) = \beta^*_{i-1} v_{i-1}(t-\tau_{i-1}) - \beta^*_i v_i(t-\tau_i),$$
$$\dot{y}_i(t) = v_i(t),$$

where $\beta^*_i = \alpha_i(\dot{x}_0)^m/(b_i)^l > 0$ is the equilibrium coefficient of the i^{th} vehicle. Here, the variables v_0, τ_0, β^*_0 are introduced for notational brevity, and set to zero. Note that, in the vicinity of the equilibrium, stability of the relative velocity (v_i) does not depend on y_i. Hence, we may drop the variables $\{y_i\}_{i=1}^N$. The resulting linearized CCFM may be succinctly written in matrix form as

$$\dot{\mathbf{V}}(t) = \sum_{i=0}^{N} A_i \mathbf{V}(t-\tau_i), \tag{2}$$

where $\mathbf{V}(t) = [v_1(t)\, v_2(t) \cdots v_N(t)]$ is the state vector at time t, and the *dynamics matrices* are given by

$$(A_0)_{kj} = \begin{cases} -\beta^*_1, & k=j=1, \\ 0, & \text{otherwise}, \end{cases}$$

$$(A_i)_{kj} = \begin{cases} -\beta^*_i, & k=j=i, \\ \beta^*_{i-1}, & k=i, j=i-1, \\ 0, & \text{otherwise}. \end{cases} \quad \text{for } 1 \le i \le N,$$

2.3 The Modified Optimal Velocity Model (MOVM)

The Modified Optimal Velocity Model (MOVM) (Kamath et al., 2017) is a recent variant of the Optimal Velocity Model (OVM) (Bando et al., 1998). The evolution equations for the MOVM, that capture the dynamics of a platoon traveling on straight road, are given by

$$\dot{v}_i(t) = a\big(\mathcal{V}(y_{i-1}(t-\tau_{i-1})) - \mathcal{V}(y_i(t-\tau_i)) - v_i(t-\tau_i)\big),$$
$$\dot{y}_i(t) = v_i(t), \tag{3}$$

for $i \in \{1,2,\ldots,N\}$. Here, $a > 0$ is the common sensitivity coefficient of the drivers, and $\tau_i \ge 0$ is the reaction delay of the i^{th} driver and we define and set v_0, τ_0, y_0 to zero for notational brevity. Further, $\mathcal{V} : \mathbb{R}_+ \to \mathbb{R}_+$ is the Optimal Velocity Function (OVF) that decides the velocity a vehicle must aim to achieve, given the headway. In the literature, an OVF is commonly assumed to satisfy the following axioms:

(i) Monotonic increase,
(ii) Bounded above, and,
(iii) Continuous differentiability.

Consequently, an OVF is invertible. For some commonly used OVFs, see (Kamath et al., 2017) and references.

The MOVM too is described by a system of non-linear delay differential equations. Similar to the CCFM, we conduct a local analysis. To that end, note that $v^*_i = 0$ and $y^*_i = \mathcal{V}^{-1}(\dot{x}_0)$ is the *unique* equilibrium for the MOVM. Linearizing system (3) about the said equilibrium, we obtain the evolution equation in matrix form as

$$\dot{\mathbf{S}}(t) = \sum_{i=0}^{N} B_i \mathbf{S}(t-\tau_i), \tag{4}$$

where $\mathbf{S}(t) = [v_1(t) \cdots v_N(t)\, u_1(t) \cdots u_N(t)]$ is the state vector at time t, with $u_i(t) = y_i(t) - y^*_i$ being the deviation of the i^{th} vehicle's headway from its equilibrium. The dynamics matrices are given by

$$B_0 = \begin{bmatrix} 0_{N\times N} & 0_{N\times N} \\ I_{N\times N} & 0_{N\times N} \end{bmatrix},$$

$$(B_i)_{kj} = \begin{cases} -a, & k=j=i, \\ -d, & k=i, j=N+i, \\ d, & k=i+1, j=i, \\ 0, & \text{elsewhere}, \end{cases} \quad \text{for } 1 \le i \le N-1$$

$$(B_N)_{kj} = \begin{cases} -a, & k=j=N, \\ -d, & k=N, j=2N, \\ 0, & \text{elsewhere}, \end{cases}$$

where $0_{N\times N}$ and $I_{N\times N}$ denote zero and identity matrices of order $N \times N$ respectively, and $d = a\mathcal{V}'(\mathcal{V}^{-1}(\dot{x}_0))$. Note that $\mathcal{V}'$ denotes derivative with respect to a state variable.

3. STRING STABILITY

In this section, we derive conditions for string stability of the CCFM and the MOVM. As mentioned in the Introduction, pairwise stability of vehicles generally does not guarantee that disturbances in state variables do not amplify down the platoon. Thus, we need to also ensure that the platoon remains *string stable* (or chain stable (Sipahi and Niculescu, 2008)/stable-over-cars (Zhang and Jarrett, 1997)). We make use of spectral-domain techniques; specifically, we derive conditions to ensure that the magnitude-squared Bode plot remains bounded above by unity for all frequencies (Sipahi and Niculescu, 2008).

3.1 The CCFM

We begin with the closed-loop pairwise dynamics of the CCFM. Individual equations in (2) are of the form

$$\dot{v}_i(t) = \beta^*_{i-1} v_{i-1}(t-\tau_{i-1}) - \beta^*_i v_i(t-\tau_i), \tag{5}$$

for $i \in \{1,2,\cdots,N\}$. Applying Laplace transform, and simplifying, we obtain the transfer function for pairwise interactions as

$$H_i(s) = \frac{V_i(s)}{V_{i-1}(s)} = \frac{\beta^*_{i-1} e^{-s\tau_{i-1}}}{s + \beta^*_i e^{-s\tau_i}},$$

for each i. Substituting $s = j\omega$ in the above equation and simplifying, we obtain

$$|H_i(j\omega)|^2 = \frac{(\beta^*_{i-1})^2}{\omega^2 + (\beta^*_i)^2 - 2\omega\beta^*_i \sin(\omega\tau_i)}. \tag{6}$$

For the CCFM to be string stable, we require $|H_i(j\omega)|^2 \le 1\ \forall\, \omega \ge 0$, for each i. Note that $\sin(\omega\tau_i) \ge -1$ for any choice of ω and τ_i. Hence, we have

$$|H_{l_i}(j\omega)|^2 \triangleq \frac{(\beta^*_{i-1})^2}{(\omega + \beta^*_i)^2} \le |H_i(j\omega)|^2\ \forall\, \omega \ge 0.$$

If we can ensure that $|H_{l_i}(j\omega)|^2 \le 1\ \forall\, \omega \ge 0$, we obtain a necessary condition for string stability of the CCFM. To that end, note that $|H_{l_i}(j\omega)|^2$ is a decreasing function in ω, with the maxima occurring at $\omega = 0$. Therefore, it suffices to ensure $|H_{l_i}(0)|^2 \le 1$. This yields the condition $\beta^*_{i-1} \le \beta^*_i$, for each i. That is, the CCFM cannot be string stable if $\beta^*_{i-1} > \beta^*_i$ for some i. Therefore, to obtain a condition for string stability of the CCFM, we henceforth impose the necessary condition $\beta^*_{i-1} \le \beta^*_i$, for each i. Therefore, from (6), an upper bound for $|H_i(j\omega)|^2$ is

$$|H_i(j\omega)|^2 \le \frac{(\beta^*_i)^2}{\omega^2 + (\beta^*_i)^2 - 2\omega\beta^*_i \sin(\omega\tau_i)}. \tag{7}$$

For a transfer function of the form depicted by the Right Hand Side (RHS) above, the necessary and sufficient condition for string stability was derived in (Sipahi and Niculescu, 2008) to be

$$\beta_i^* \tau_i \leq \frac{1}{2} \ \forall\ i.$$

However, (7) represents an upper bound on the magnitude-squared Bode plot of the CCFM. Therefore, a sufficient condition for string stability of the CCFM is,

$$\beta_{i-1}^* \leq \beta_i^*, \text{ and } \beta_i^* \tau_i \leq \frac{1}{2} \ \forall\ i. \tag{8}$$

Remark 1. The inequalities in (8) may be interpreted as imposing a platoon-wide and a pairwise constraint respectively. That is, the first inequality implies that the equilibrium coefficients must be non-increasing in the vehicle index (platoon-wide constraint), while the second inequality provides a bound on parameters of individual vehicles (pairwise constraint).

Remark 2. For the uniform traffic flow (*i.e.,* $b_i = b\ \forall i$), the platoon-wide condition can be interpreted as follows. Vehicles farther away from the lead vehicle must be able to accelerate/decelerate faster (larger values of α_i) to ensure string stability.

Remark 3. Condition (8) generalizes the result in (Sipahi and Niculescu, 2008) to transfer functions of the form (6).

3.2 The MOVM

We begin with the closed-loop pairwise dynamics of the MOVM. On combining the i^{th} and the $(N+i)^{th}$ equations from (4), we obtain

$$\ddot{u}_i(t) = du_{i-1}(t - \tau_{i-1}) - du_i(t - \tau_i) - av_i(t - \tau_i).$$

On applying Laplace transform, we obtain

$$H_i(s) = \frac{U_i(s)}{U_{i-1}(s)} = \frac{de^{-s\tau_{i-1}}}{s^2 + (as + d)e^{-s\tau_i}},$$

for each i. Substituting $s = j\omega$ and simplifying, we obtain the magnitude-squared frequency response as

$$|H_i(j\omega)|^2 = \frac{d^2}{\alpha(\omega) + d^2},$$

where $\alpha(\omega) = \omega^2(\omega^2 - 2aw\sin(\omega\tau_i) - 2d\cos(\omega\tau_i) + a^2)$. Note that finding a condition for $|H_i(j\omega)|^2 \leq 1\ \forall\omega \geq 0$ is equivalent to finding a condition for $\alpha(\omega) \geq 0\ \forall\omega \geq 0$. Since ω^2 is always non-negative, it suffices to find a condition such that $\omega^2 - 2aw\sin(\omega\tau_i) - 2d\cos(\omega\tau_i) + a^2 \geq 0\ \forall\omega \geq 0$ holds. Finding such a condition seems to be analytically intractable. Hence, we now derive a sufficient condition for string stability of the MOVM as follows. Note that $\omega\tau_i \geq \sin(\omega\tau_i)$ and $\cos(\omega\tau_i) \leq 1\ \forall\omega, \tau_i \geq 0$. Substituting these in the above, we find a condition such that $\omega^2 - 2a\tau_i w^2 - 2d + a^2 \geq 0$ holds $\forall\omega \geq 0$. That is, we need to find a condition such that $(1 - 2a\tau_i)\omega^2 + (a^2 - 2d) \geq 0\ \forall\omega \geq 0$. Geometrically, this condition mandates that the described parabola lies above the ω-axis. To satisfy this, we require $1 - 2a\tau_i > 0$ (the parabola faces above) and $a^2 - 2d \geq 0$ (the vertex of the parabola lies above the ω-axis). Combining these conditions, we state a sufficient condition for string stability of the MOVM: for each i, we require

$$a^2 \geq 2d \text{ and } a\tau_i < \frac{1}{2}. \tag{9}$$

Remark 4. Similar to Remark 1, the first inequality in the above condition may be thought as imposing a platoon-wide constraint, while the second inequality enforces a pairwise constraint.

4. ROBUST STABILITY

In this section, we consider parametric uncertainties in the CCFM and the MOVM. Specifically, we assume that the model parameters other than the reaction delay lie in an interval. We then derive bounds on the reaction delay such that the CCFM and the MOVM will remain locally asymptotically stable for any value of the remaining parameters. Specifically, we consider two types of robust stability; namely, (a) pairwise and (b) platoon.

4.1 The CCFM

The characteristic equation for the CCFM is given by (Kamath et al., 2015, Equation (8))

$$s + \beta_i^* e^{-s\tau_i} = 0. \tag{10}$$

Note that this is of the form $p(s) + q(s)e^{-s\tau_i} = 0$, where $p(s) = s$ and $q(s) = \beta_i^*$. Here, $\deg(p) > \deg(q)$, and $p(s)$ is a stable polynomial. Therefore, the necessary and sufficient condition for the CCFM to be robust stable *independent of the delay* is $|p(j\omega)| > |q(j\omega)|\ \forall\omega \geq 0$ (Kharitonov, 1999, Section 3). This simplifies to $\beta_i^*/\omega < 1\ \forall\omega \geq 0$. Clearly, this is not true. Hence, the CCFM cannot be robust stable independent of the delay. Therefore, we next derive conditions for delay-dependent robust stability. Specifically, we consider that the equilibrium coefficient realizes as $\beta_i^* \in [\underline{\beta}_i^*, \overline{\beta}_i^*]$, for each i.

Delay-dependent pairwise robust stability: For systems with characteristic equations of the form (10), a sufficient condition for pairwise stability is $\beta_i^*\tau_i < 1$ (Kharitonov and Melchor-Aguilar, 2000, Lemma 3). Therefore, a sufficient condition for delay-dependent pairwise robust stability of the CCFM is that, for each i, we require

$$\overline{\beta}_i^* \tau_i < 1. \tag{11}$$

Delay-dependent platoon robust stability: To obtain a sufficient condition for delay-dependent platoon robust stability, we consider the evolution of the overall platoon captured by (2), instead of pairwise interactions as done in Section 4.1.1. For systems of this form, a sufficient condition for local stability is $\sum_{i=0}^{N} \|A_i\|\tau_i < 1$ (Kharitonov and Melchor-Aguilar, 2000, Theorem (6)). Using the *Frobenius norm* of a matrix, we obtain a sufficient condition for delay-dependent platoon robust stability of the CCFM as

$$\sum_{i=1}^{N} \overline{\beta}_i^* \tau_i < 1. \tag{12}$$

4.2 The MOVM

We now derive the corresponding conditions for the MOVM. We assume that the parameters realize as $a \in [\underline{a}, \overline{a}]$ and $d \in [\underline{d}, \overline{d}]$.

Delay-dependent pairwise robust stability: Pairwise interactions for the MOVM are captured by the following evolution equations

$$\begin{bmatrix}\dot{v}_i(t)\\ \dot{u}_i(t)\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}v_i(t)\\ u_i(t)\end{bmatrix} + \begin{bmatrix}-a & -d\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_i(t-\tau_i)\\ u_i(t-\tau_i)\end{bmatrix}$$

Therefore, applying (Kharitonov and Melchor-Aguilar, 2000, Theorem (6)) with the Frobenius norm, a sufficient condition for delay-dependent pairwise robust stability of the MOVM is that, for each i, we require

$$\tau_i\sqrt{\overline{a}^2 + \overline{d}^2} < 1. \tag{13}$$

Delay-dependent platoon robust stability: The evolution equations of the MOVM are given by (4). For systems of this form, a sufficient condition for local stability is $\sum_{i=0}^{N} \|B_i\|\tau_i < 1$ (Kharitonov and Melchor-Aguilar, 2000, Theorem (6)). Using the Frobenius norm of a matrix, a sufficient condition for delay-dependent platoon robust stability of the MOVM can be shown to be

$$\left(\sqrt{\overline{a}^2 + \overline{d}^2}\right)\sum_{i=1}^{N}\tau_i < 1. \tag{14}$$

Remark 5. From (11) to (14), it is clear that conditions for platoon robust stability are relatively stringent in comparison to their pairwise counterparts. Moreover, platoon robust stability can be considered as a robust version of string stability.

5. COMPARISON OF RESULTS

In this section, we compare our conditions for string stability and robustness derived in the previous sections with those of local pairwise stability and non-oscillatory condition derived in (Kamath et al., 2016) for the CCFM and in (Kamath et al., 2017) for the MOVM.

For the CCFM, we assume that $\beta_i^* = \beta^*\ \forall i$, for ease of comparison. Thus, (8) represents the necessary and sufficient condition for string stability. Further, for pairwise robust stability captured by (11), we use β^* as a proxy for $\overline{\beta}^*$ to facilitate comparison. That is, we assume that the parameter is fixed at "worst possible" value. Finally, the CCFM will be locally pairwise stable if and only if (Kamath et al., 2016, Equation (21))

$$\beta^*\tau_i < \frac{\pi}{2}\ \forall i. \tag{15}$$

Let $\mathcal{S}_{k_i}$ denote the set of all reaction delay values of the i^{th} vehicle that satisfy property $k \in \{LS, SS, RS\}$, where LS : local pairwise stability, SS : string stability and RS : pairwise robust stability. Then, from equations (8), (11) and (15), we note that: $S_{SS_i} \subset S_{RS_i} \subset S_{LS_i}$ for each i. In other words, when the reaction delay of a given vehicle satisfies the condition for string stability of the CCFM, it automatically satisfies other criteria as well.

We now summarize the results for the MOVM; it will be

(i) locally pairwise stable if and only if

$$\chi\tau_i < \tan^{-1}\left(\frac{\chi}{\tilde{d}}\right)\ \forall i, \tag{16}$$

$$\text{where } \chi = \sqrt{\frac{a(a+\sqrt{a^2+4\tilde{d}^2})}{2}}.$$

(ii) string stable if

$$a^2 \geq 2d \text{ and } a\tau_i < \frac{1}{2}\ \forall i.$$

(iii) pairwise robust stable if

$$\tau_i\sqrt{a^2+d^2} < 1\ \forall i.$$

We now make use of a *stability chart* to understand the inclusion of various stability conditions. To that end, recall from Section 2 that $d = a\mathcal{V}'(\mathcal{V}^{-1}(\dot{x}_0))$. Denote $\mathcal{V}'(\mathcal{V}^{-1}(\dot{x}_0))$ by $\tilde{d}$. Then, $d = a\tilde{d}$. Note that $\tilde{d}$ depends on the OVF and the cruise velocity of the lead vehicle. For the case of pairwise robust stability, we assume that a is fixed. We also use $\tilde{d}$ as a proxy to denote $\overline{\tilde{d}}$.

To obtain the stability chart, we fix $a = 4$ and vary $\tilde{d}$ in the interval $[1, 2]$. Note that this variation conforms with the platoon-wide constraint $a^2 \geq 2d$ imposed by string stability. We then compute various bounds on the reaction delay using the scientific computation tool MATLAB; we denote by τ_k the bound on the reaction delay for property $k \in \{LS, SS, RS\}$, where LS : local pairwise stability, SS : string stability and RS : pairwise robust stability.

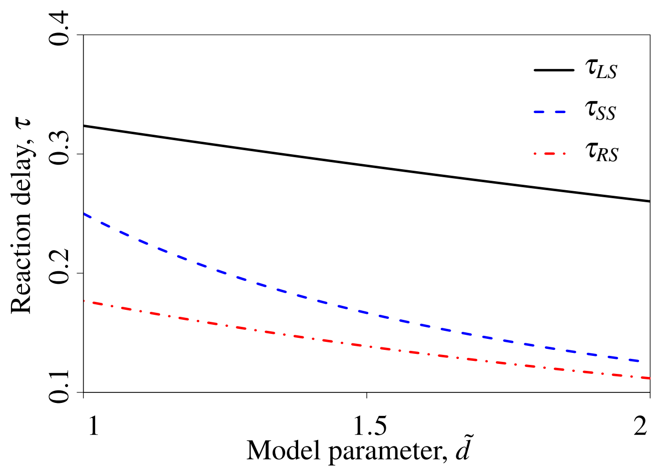

Fig. 1. *Stability chart:* Diagram depicting the boundaries of local pairwise stability (τ_{LS}), string stability (τ_{SS}) and pairwise robust stability (τ_{RS}) for the MOVM.

Figure 1 portrays the ensuing stability chart. As can be seen from this diagram, if the reaction delay satisfies the condition for pairwise robust stability, it satisfies the remaining conditions as well.

Remark 6. From equations (8), (9), (11), (13), (15) and (16), we note that pairwise stability (both, local and robust) and string stability can be ensured in a decentralized manner using only on-board sensors. This is because each vehicle requires information from the vehicle directly ahead only. However, as seen from (12) and (14), ensuring platoon robust stability requires each vehicle to be equipped with advanced communication devices.

6. CONCLUDING REMARKS

In this paper, we studied two important properties of connected vehicle systems; namely, string stability and robustness to parametric uncertainties. Specifically, we considered a platoon of vehicles traveling on a straight road with no overtaking. We modeled the dynamics of individual vehicles by the Classical Car-Following Model (CCFM) and the Modified Optimal Velocity Model (MOVM).

First, we derived conditions for string stability of both models using spectral-domain techniques. Specifically, for the CCFM, we first derived a necessary condition for string stability. When this condition holds, we derived a sufficient condition for string stability of the CCFM, which generalizes known results in the literature. For the MOVM, we derived a sufficient condition for string stability. To the best of our knowledge, such a result is the first of its kind. Our analyses clearly bring forth the platoon-wide and pairwise constraints imposed by the derived conditions.

Next, we derived conditions for both models to be robust to parametric uncertainties. In particular, we derived bounds on the reaction delay when the remaining parameters lie in an interval. For both models, we derived a sufficient condition each for (a) pairwise robust stability, and (b) platoon robust stability.

Finally, we compared our results with necessary and sufficient conditions for each model to remain locally pairwise stable. For the CCFM, we inferred that string stability imposes the most stringent conditions on the reaction delay, while for the MOVM, it is pairwise robust stability.

In addition to possibly providing design guidelines for connected vehicle systems, our analyses suggest the possibility of achieving pairwise (both, local and robust) and string stabilities in a decentralized manner; *i.e.,* with only onboard sensors. However, for the entire platoon of connected vehicles to be robust stable, each vehicle should be installed with additional communication devices.

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