Noise-induced Dynamics in a Delayed Triple-well Potential System Driven by Correlated Noises

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Abstract: In this paper, the effects of time delay and correlated noises on noise-induced dynamics in a delayed triple-well potential system are studied. Using the linear response theory, we derive the expression of the transient response to show the effectiveness of the system response to a periodic forcing. The effectiveness of the system response to an external forcing can be improved in a triple-well system by choosing the proper time delay and noise cross-correlation. Moreover, the power spectrum and the quality factor are calculated to quantify coherence resonance (CR). It is found that the power spectrum and the quality factor show the peak structure as the optimal noise intensities are chosen. That is, the CR appears in this system. The noise cross-correlation can break the symmetry of the triple-well potential and induce the transition of the interwell resonance among different wells. The presence of time delay in the triple-well potential enhances the role of multiplicative noise in the regularity dynamics.

Keywords: Delayed triple-well potential, transient response, coherence resonance, correlated noises, quality factor.

1. INTRODUCTION

Time delay exists in a wide variety of natural and manmade systems, such as laser physics, biological systems, mechanical and electrical systems. Most dynamical systems with time delays can be modelled as the delay-differential equations (DDEs), which are infinite-dimensional systems with an infinite number of initial conditions. Over the past decades, a great progress has been made for DDEs in both the theoretical methods and practical applications. Among them, we refer the readers to the classic books written by Hale (1977), Qin et al. (1989), Stépán (1989), Diekmann et al. (1995) and Hu et al. (2002).

In practice, the random fluctuation or environmental noise is an unavoidable factor existing in system modelling and may lead to the discovery of some counterintuitive phenomena. Particularly, stochastic resonance (SR) is a phenomenon in which the optimal noise intensity results in a maximal response of the dynamical system to a weak input signal, and coherence resonance (CR) describes the increased regularity of the output in an excitable system by the addition of moderate noise intensity. Therefore, the effects of noise on the DDEs have attracted the attentions of scientists from many fields. For example, Ohira et al. (1999) illustrated the resonance behavior between noise and delay both numerically and analytically in a two-state model. Guillouzic et al. (1999, 2000) proposed a small delay approximation method and applied it to a stochastic delayed differential equation (SDDE) with non-delayed diffusion. Tsimring et al. (2001) established the theory of a bistable system with noise and time delay within the framework of the two-state system approximation. Masoller (2002) found that the appearance of resonant behavior is due to the interplay of noise and delayed feedback in a single-mode semiconductor laser. Jin et al. (2007, 2012, and 2015) studied the noise-induced resonances and the delay-independent stability of the delayed bistable and linear systems. However, most of the previous studies focus on linear and bistable systems, only a few publications involve the multi-stable systems.

Multi-stable dynamical systems mean the coexistence of several possible attractors for a given set of parameters. Pisarchik et al. (2014) pointed out the coexistence of different stable states offers a great flexibility in the system performance without major parameter changes. Therefore, the research of noise-induced dynamics in the multi-stable systems is important because the phenomenon of multi-stability exists in many fields, such as biological systems (see Foss et al. (1996, 1997, and 2000)), energy harvesting system (see Zhou et al. (2014)), optical systems (see Brambilla et al. (1991)) and social systems (see Sneppen et al. (2012)). For example, Ma et al. (2007) studied the multistability in spiking neuron models of delayed recurrent inhibitory loops. Campbell et al. (1995) demonstrated the existence of limit cycles, two-tori, and multistability in a damped harmonic oscillator with delayed negative feedback. Later, Nicolis et al. (2017) analyzed how the phenomenon of SR is modulated as a triple-well system is moved across its bifurcation diagram. Jia (2009) investigated the effects of the time delay on the stationary properties of a triple-well system driven by the correlated noises. Jin et al. (2017) explored the CR and SR in a periodic potential system driven by multiplicative and additive noises. Xu et al. (2017, 2018) studied the SR in a delayed triple-well potential and a couple system with four-well potential, respectively. To the best knowledge of authors, less attention has been paid to the stochastic response and CR in a delayed triple-well potential with correlated noises.

The aim of this paper is to study the transient response and CR in a delayed triple-well potential system driven by correlated multiplicative and additive noises. In Section 2, the expressions of transient response are obtained by using the linear response theory. The effects of time delay and noise cross-correlation on the stochastic response to a periodic forcing in the triple-well potential system are analyzed. In Section 3, CR are explored by using the quantifiers, such as the power spectrum and the qualify factor. The influences of noise intensities and time delay on CR are discussed. Some conclusions are drawn in Section 4.

2. STOCHASTIC RESPONSE

2.1 The Model

The system of concern is an over-damped particle moving in a delayed triple-well potential, which is driven by correlated noises and a periodic forcing:

$$\begin{aligned}\dot{x} = -[ax_\tau^5 - b(1+h)x^3 + hx] + \varepsilon \sin(\omega t) \\ + x\xi(t) + \eta(t),\end{aligned} \tag{1}$$

where $x_\tau = x(t-\tau)$ and τ is the time delay, ε and ω represent the amplitude and the frequency of a periodic forcing, respectively. a, b, and h are the parameters of the potential function. The multiplicative noise $\xi(t)$ and additive noise $\eta(t)$ are cross-correlated Gaussian white noises with zero mean and Dirac δ correlation functions as follows:

$$\begin{aligned}&\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'),\ \langle \eta(t)\eta(t')\rangle = 2Q\delta(t-t'),\\ &\langle \xi(t)\eta(t')\rangle = \langle \xi(t')\eta(t)\rangle = 2\lambda\sqrt{DQ}\delta(t-t').\end{aligned} \tag{2}$$

where D and Q are the multiplicative and additive noise intensity, respectively. λ is the cross-correlation between multiplicative and additive noises. The triple-well potential with $\tau = 0$, i.e. $V(x) = ax^6/6 - b(1+h)x^4/4 + hx^2/2$ for fixed $a = 1/30$, $b = 1/5$ and $h = 3/10$ is plotted in Fig. 1. It is clear that the potential has three stable states $s_i(i = 1,2,3)$ and two unstable states $u_j(j = 1,2)$.

Using the small time delay approximation proposed by Guillouzic et al. (1999), the corresponding Fokker-Planck equation of (1) can be derived as following

$$\begin{aligned}\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}\left[\alpha(x) + \beta(x)\frac{d\beta(x)}{dx}\right]p(x,t) \\ + \frac{\partial^2}{\partial x^2}\beta^2(x)p(x,t),\end{aligned} \tag{3}$$

where $\alpha(x) = [-ax^5 + b(1+h)x^3 - hx + \varepsilon\sin(\omega t)]\rho(x)$,

$\beta(x) = [Dx^2 + 2\lambda(DQ)^{1/2}x + Q]^{1/2}\rho(x)$, $\rho(x) = 1 + 5a\tau x^4$.

By setting the left side of (3) to zero, the generalized potential $\tilde{V}(x,t)$ can be given by

$$\tilde{V}(x,t) = V_0(x) - \varepsilon g(x)\sin(\omega t), \tag{4}$$

where $V_0(x) = D\int^x \rho(u)[au^5 - b(1+h)u^3 + hu]/\beta^2(u)\,du$,

$g(x) = \int^x D\rho(u)/\beta^2(u)\,du$.

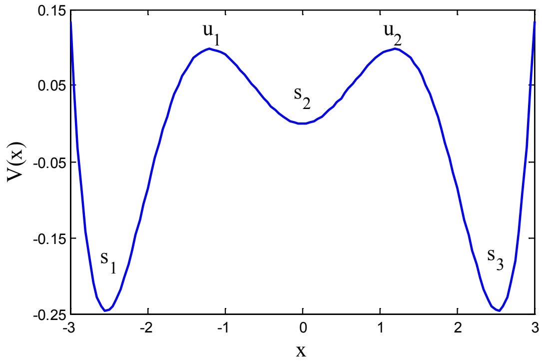


Fig. 1. Plot of the triple-well potential V(x).

2.2 Transient Response

The effectiveness of the system response to a periodic forcing is investigated through the transient response. For small signal amplitude ε, the transition probability masses p_i $(i = 1,2,3)$ between the attraction basins of the stable states s_i have the form $\mathbf{P} = \mathbf{P}^{(0)} + \varepsilon\Delta\mathbf{P}$. The $\Delta\mathbf{P}$ satisfies the following equations:

$$\frac{\mathrm{d}\Delta\mathbf{P}}{\mathrm{d}t} = \mathbf{W}\Delta\mathbf{P} + \boldsymbol{\varphi}\sin(\omega t). \tag{5}$$

where $\mathbf{P} = (p_1 \quad p_2 \quad p_3)^T$, $\mathbf{P}^{(0)} = (p_1^{(0)} \quad p_2^{(0)} \quad p_3^{(0)})^T$ the superscript T denotes the transpose of matrix. And

$$\mathbf{W} = \begin{pmatrix} -W_{1,2} & W_{2,1} & 0 \\ W_{1,2} & -(W_{2,1} + W_{2,3}) & W_{3,2} \\ 0 & W_{2,3} & -W_{3,2} \end{pmatrix}.$$

Here the components of $\mathbf{W}$ are obtained from (4) by using the adiabatic approximation:

$$W_{m,m+1} = \frac{\sqrt{V_0''(s_m)\left|V_0''(u_m)\right|}}{2\pi}\exp\left\{\frac{V_0(s_m) - V_0(u_m)}{D}\right\}, (m = 1,2).$$

$$W_{n,n-1} = \frac{\sqrt{V_0''(s_n)\left|V_0''(u_{n-1})\right|}}{2\pi}\exp\left\{\frac{V_0(s_n) - V_0(u_{n-1})}{D}\right\}, (n = 2,3).$$

The $\boldsymbol{\varphi}$ in (5) is expressed as follows:

$$\boldsymbol{\varphi} = \frac{1}{D}\begin{pmatrix} -W_{1,2}\Delta g_{1,2}p_1^{(0)} + W_{2,1}\Delta g_{2,1}p_2^{(0)} \\ W_{1,2}\Delta g_{1,2}p_1^{(0)} - (W_{2,1}\Delta g_{2,1} + W_{2,3}\Delta g_{2,3})p_2^{(0)} + W_{3,2}\Delta g_{3,2}p_3^{(0)} \\ W_{2,3}\Delta g_{2,3}p_2^{(0)} - W_{3,2}\Delta g_{3,2}p_3^{(0)} \end{pmatrix}.$$

(6)

where $\Delta g_{m,m+1}=g(u_m)-g(s_m)$, $\Delta g_{n,n-1}=g(u_{n-1})-g(s_n)$, $(m=1,2;n=2,3)$.

Setting the matrix $\mathbf{G}=(\boldsymbol{\xi}_1,\ \boldsymbol{\xi}_2,\ \boldsymbol{\xi}_3)$ ($\boldsymbol{\xi}_i$ are the eigenvectors of $\mathbf{W}$) and the variables $\mathbf{a}=\mathbf{G}^{-1}\Delta\mathbf{p}$, one obtains the following equations from (5):

$$\frac{\mathrm{d}a_i(t)}{\mathrm{d}t}=\gamma_i a_i+\sum\nolimits_{j=1}^{3}\mathbf{G}^{-1}{}_{ij}\ \boldsymbol{\varphi}_j\ \sin(\omega t), \qquad (7)$$

where γ_i $(i=1,2,3)$ stand for the eigenvalues of $\mathbf{W}$.

The transient response $\Delta p_i(t)$ in (5) can be obtained by solving (7) as follows:

$$\Delta p_i(t)=\sum\nolimits_{j=1}^{3}\mathbf{G}_{i,j}\exp(\gamma_j t)$$
$$\cdot\int_0^t\sum\nolimits_{k=1}^{3}\mathbf{G}^{-1}{}_{jk}\ \boldsymbol{\varphi}_k\ \sin(\omega u)\exp(-\gamma_j u)du \quad (8)$$

For the unperturbed system the probability masses in middle well reaches half of its asymptotic value at time $t_{1/2}^{(0)}$, i.e.,

$$p_2^{(0)}(t_{1/2}^{(0)})=0.5\lim_{t\to\infty}p_2^{(0)}(t_{1/2}^{(0)}). \qquad (9)$$

If $\Delta p_2(t_{1/2}^{(0)})>0$, the periodic forcing would be deemed effective. Then, the system would accelerate the crossing of level 1/2 of the full response $p_2(t)=p_2^{(0)}(t)+\varepsilon\Delta p_2(t)$, which is a primary indicator of the effectiveness of the periodic forcing (see Nicolis (2012)).

The unperturbed $p_2^{(0)}(t)$ is derived with the initial state $p_1^{(0)}(0)=1$ and $p_2^{(0)}(0)=p_3^{(0)}(0)=0$ as

$$p_2^{(0)}(t)=\frac{1}{m_1}\Big\{m_2\Big[1-\cosh\left(0.5\sqrt{m_5}t\right)\exp\left(-0.5m_3t\right)\Big]+ \qquad (10)$$
$$\Big[\sinh\left(0.5\sqrt{m_5}t\right)\exp\left(-0.5m_3t\right)m_4\Big]\Big/\sqrt{m_5}\Big\},$$

where

$\lim\limits_{t\to\infty}p_2^{(0)}(t)=m_2/m_1$, $m_1=W_{12}W_{32}+W_{21}W_{32}+W_{23}W_{12}$,

$m_2=W_{12}W_{32}$, $m_3=W_{12}+W_{21}+W_{23}+W_{32}$,

$m_4=W_{12}\left[W_{21}W_{32}-W_{23}W_{32}-W_{32}^2+W_{12}W_{32}+2W_{12}W_{23}\right]$,

$m_5=W_{12}^2+W_{21}^2+W_{23}^2+W_{32}^2+2W_{23}W_{32}-2W_{23}W_{12}$
$\quad+2W_{21}W_{23}-2W_{12}W_{32}-2W_{21}W_{32}+2W_{12}W_{21}$.

2.3 Analysis of Stochastic Response

For the case of two correlation noises, the time $t_{1/2}^{(0)}$ in (9) is too complicated to present through (10), so the numerical algorithm is adopted to obtain $t_{1/2}^{(0)}$. Particularly, for two independent noises the formula $t_{1/2}^{(0)}=\ln(2)\big/\left(W_{12}+2W_{21}\right)$ is satisfied. Furthermore, the transient response in middle well at time $t_{1/2}^{(0)}$ i.e., $\Delta p_2(t_{1/2}^{(0)})$ is derived from (8).

In Figs. 2(a) and 2(b), the transient response $\Delta p_2(t_{1/2}^{(0)})$ as a function of driving frequency ω is displayed for different values of time delay τ and cross-correlation λ , respectively. Note that to guarantee the condition of adiabatic approximation, ω is limited to the range of $\omega<<V_0^{''}(S_i)$ (see Nicolis et al. (2017)). In Fig. 2(a), the range of ω corresponding to the positive value of $\Delta p_2(t_{1/2}^{(0)})$ is almost the same length for different τ . But, $\Delta p_2(t_{1/2}^{(0)})$ decreases with an increase in τ . That is, the increasing time delay leads to the weakening of the transient response to a periodic forcing in the low frequency region. Moreover, it can be seen from Fig. 2(b) that $\Delta p_2(t_{1/2}^{(0)})$ increases as the increment in λ when $\omega<0.055$. However, when $\omega>0.055$, $\Delta p_2(t_{1/2}^{(0)})$ shows a non-monotonic change with λ. In other words, for a given driving frequency there is an optimal λ that would maximize the transient response of the system. The τ and λ play an opposite role in the enhancement of transient response for the low frequency forcing, which is different from the case of asymptotic response shown in the work by Xu et al (2017). Thus, one can improve the effectiveness of the system response to a periodic forcing in the triple-well system by controlling the time delay and noise cross-correlation.

If the periodic forcing $\varepsilon\sin(\omega t)$ in (1) is replaced by $\varepsilon\sin(\omega t+\theta)$ with non-zero phase θ, the transient response $\Delta p_2(t_{1/2}^{(0)})$ as a function of θ varies with period 2π shown in Fig. 2(c). As expected, the amplitude of $\Delta p_2(t_{1/2}^{(0)})$ achieves a finite value as the driving frequency ω decreases to infinitesimal. On the contrary, the system response can vanish in the limit where ω is larger than the inverse of the characteristic time of the diffusion process around each of the stable states. In particular, these values of $\Delta p_2(t_{1/2}^{(0)})$ which are negative in Figs. 2(a) and 2(b) can again be positive within an appropriate range of phase θ values; thus, the periodic forcing to the system can still be deemed effective.

3. COHERENCE RESONANCE

When the periodic forcing is absent (i.e., $\varepsilon=0$), the CR of the system (1) is characterized by the power spectrum and the quality factor in this section. The quality factor is defined by Hu et al. (1993) as $\beta=h\omega_p/\Delta\omega$, where h is the peak height of the power spectrum, ω_p represents the peak frequency and $\Delta\omega$ is the width of the spectrum measured at the height of $h/\sqrt{e}$. In the numerical simulations, the sample frequency is chosen as $Fs=100\text{Hz}$, 10^4 different realizations are performed and the data length takes 5×10^4 for each realization. Here, the power spectrum as a function of frequency is shown for different additive noise intensity Q and multiplicative noise intensity D in Fig. 3. It is clear that the peak height of the power spectrum first increases and then decreases as both Q and D increase (see Figs. 3(a) and 3(b)).

That is, the power spectrum attains the maximum at the optimal noise intensities, which indicates the possibility of an occurrence of CR phenomenon. Noting that the peak frequency in Fig. 3(b) is increased with the enlargement of D, which is different from the case of Q (see Fig. 3(a)).

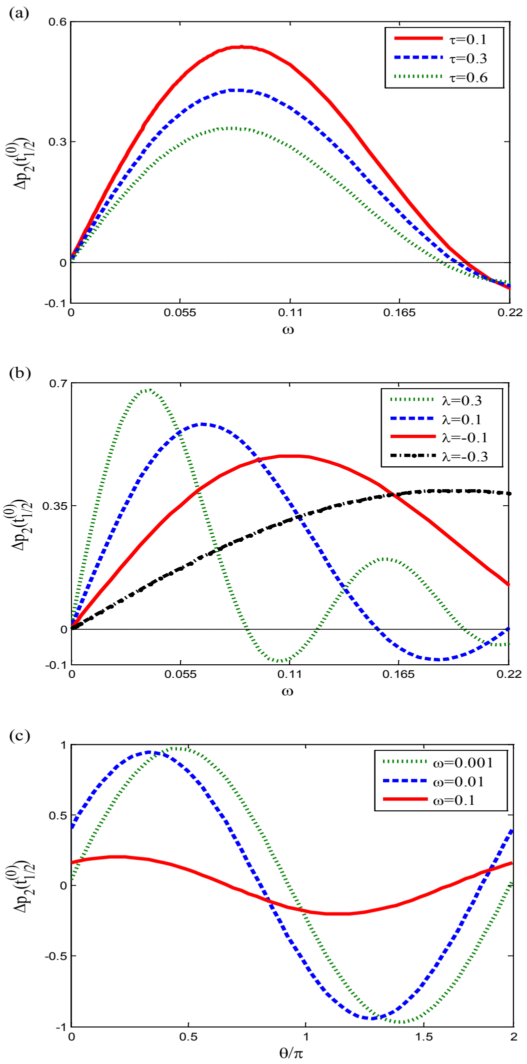
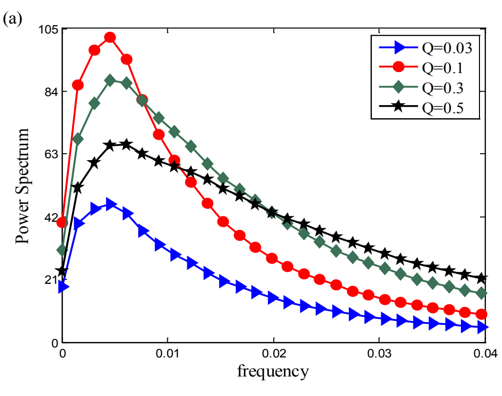


Fig. 2. *Plot of the transient response* $\Delta p_2(t_{1/2}^{(0)})$ *for (a) different* τ *with* $\lambda=0$; *(b) different* λ *with* $\tau=0.1$; *(c) different* ω *with* $\tau=0.1$, $\lambda=0.4$. *The other parameters are chosen as* $Q=D=0.05$.



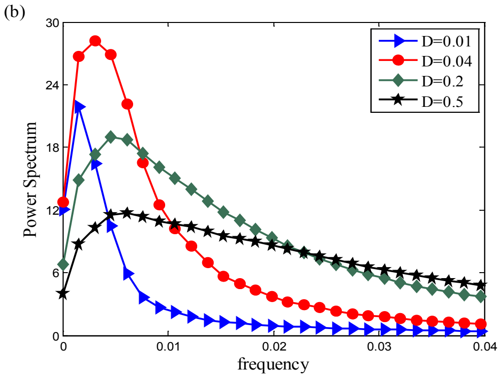


Fig. 3. The power spectrum of the system versus frequency for (a) different Q with $D=0.1$ and (b) different D with $Q=0.01$. The other parameters are chosen as $\tau=0.03$ and $\lambda=0.5$.

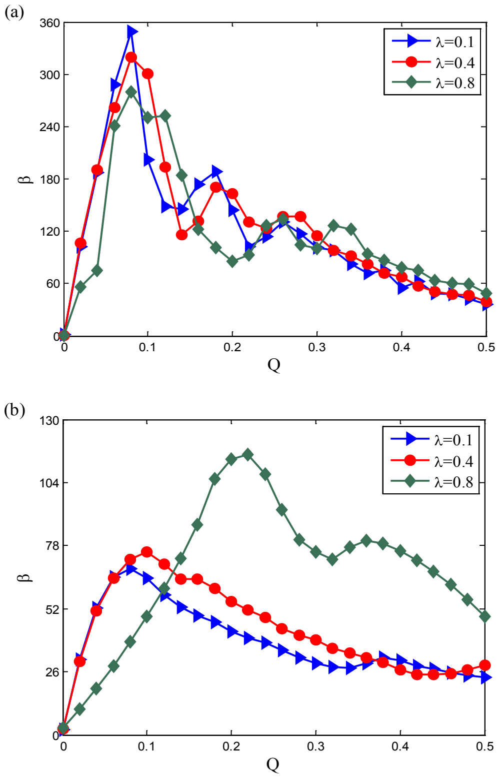


Fig. 4. The quality factor β as a function of Q for different λ with $\tau=0$: (a) $D=0.01$ and (b) $D=0.1$.

Figure 4 displays the quality factor β as a function of Q for different λ and D. In Fig. 4(a), the curve of β shows the oscillatory character for small D. The reason is that, for the triple-well potential system, the resonances in the single potential well and between the lateral potential wells are coexistent. The system mainly concentrates on the intrawell motion in the lateral potential wells owing to the small D, which induces the multi-peak phenomenon in the curve of β. With the increment in λ in Fig. 4(a), the maximal peak value of β decreases but the noise intensity that induced resonance remains almost unchanged. Especially, for sufficiently large D as shown in Fig. 4(b), when λ is increased, the peak value of CR ascends (a trend opposite to the one in Fig. 4(a)) and the position of the peak is shifted toward the direction of increasing Q. One possible explanation of this observation is that the large cross-correlation between multiplicative and additive noises strongly breaks the symmetry of the triple-well potential; thus, the increasing D or Q induces the transition of the interwell resonance from two adjacent wells to two side wells. So the CR phenomenon is quite sensitive to the correlated multiplicative and additive noises in the triple-well potential system. It demonstrates that the correlated noises can significantly affect the regularity of system dynamics.

The effects of time delay τ on CR are analyzed in Fig. 5. When τ is increased in Figs. 5(a) and 5(b), these peak values of β versus Q or D decrease. Thus, the increase of time delay results in the weakening of the regularity of the system according to the CR effect. Moreover, the additive noise intensity of resonance keeps almost fixed in Fig. 5(a) but the optimal multiplicative noise intensity is decreased by increasing the time delay. In fact, the intensity of multiplicative noise being different from additive noise does not scale with an inverse power of the system size, and the effect of multiplicative noise depends on the state of the system (see Horsthemke et al. (1983). The presence of time delay enhances the role of multiplicative noise in the regularity dynamics.

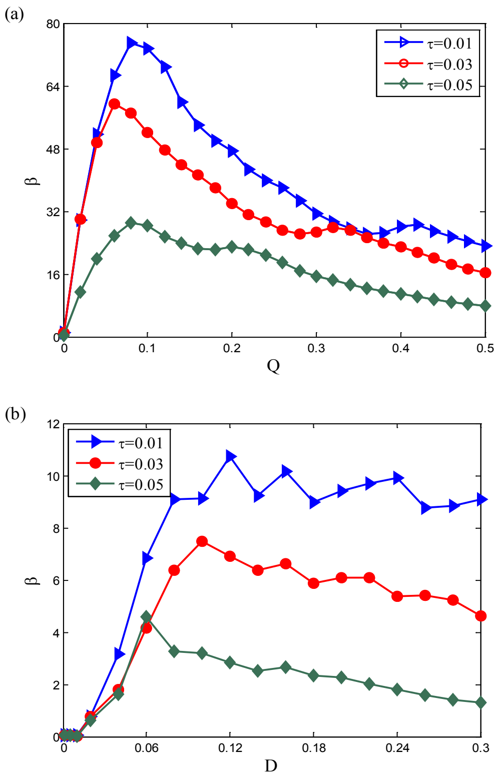


Fig. 5. Plot of the quality factor β for different τ with $\lambda = 0$ as a function of (a) Q with $D = 0.08$ and (b) D with $Q = 0.005$.

To explore the combined effects of correlated noises and time delay on CR, the quality factor β versus noise cross-correlation λ is plotted with fixed $D = 0.03$ for different time delay τ and additive noise intensity Q in Fig. 6. The curve of β versus λ shows the symmetry structure, which demonstrates the positive and negative cross-correlation has the same effect on the CR. It is obvious that the curve of β versus λ can be transferred from single peak (see Fig. 6(a)) into the collapse shape (see Fig. 6(b)) at the center of $\lambda = 0$ as Q varies from 0.03 to 0.3. For small additive noise intensity in Fig. 6(a), the system subjected to uncorrelated noises has the largest regularity than that with correlated noises. However, as Q increases to a tenfold, there is an optimal noise cross-correlation that can generate the largest regularly activity. Moreover, with the enlargement of τ, the single peak (see Fig. 6(a)) and the collapse shape (see Fig. 6(b)) become less pronounced and β attains a very shallow extremum at the value of $\lambda = 0$. Therefore, the influences of smaller noise cross-correlation on regularity are not evident in the system with larger time delay.

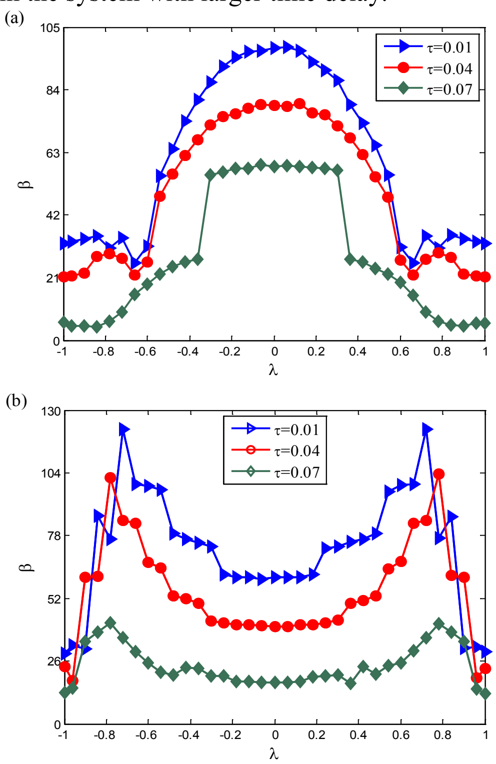


Fig. 6. The quality factor β as a function of λ for different τ with $D = 0.03$ and (a) $Q = 0.03$; (b) $Q = 0.3$.

4. CONCLUSIONS

This paper presents the analysis of the transient response and CR in a delayed triple-well potential system with correlated noises and a periodic forcing. The obtained theoretical and numerical results show that the effectiveness of the stochastic response to a periodic forcing in the triple-well potential system can be improved by adjusting the time delay and noise cross-correlation. Particularly, the combined effects of both time delay and correlated noises on CR are explored by the power spectral and the quality factor. It is found that the noise cross-correlation can break the symmetry of the triple-well potential and significantly affects the regularity of system dynamics. The CR effect can be weakened by increasing the time delay. Moreover, the random fluctuation adopted in this paper is Gaussian white noise, which has zero correlation time. When the correlation time of the fluctuation is large, we must choose a stochastic process with non-zero correlation time, such as colored noise, dichotomous noise and non-Gaussian noise, which will be considered in our future work.

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