Polynomial (Chaos) approximation of the Spectral Abscissa: Efficiency and Limitations.

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Uncertainty often affects dynamical systems. Polynomial (chaos) approximations approximate a quantity of interest as a function of uncertain parameters and extract relevant statistical information. The aim of this talk is to show how the smoothness properties of the rightmost (or dominant) eigenvalues, and in particular spectral abscissa function (a frequently used stability measure), may affect both the quality of polynomial (chaos) approximations as well as the numerical computation of statistical information.

The real part of the rightmost eigenvalue, the so-called spectral abscissa, describes the exponential growth or decay rate of the solution of a linear time delay system, inferring the stability of the system. In an engineering context, the spectral abscissa depends on physical quantities like lengths ℓ, which assume values in $\mathbb{S} \subset \mathbb{R}^D$, and is expressed by the function $\ell \mapsto \alpha(\ell)$, where

$$\alpha(\ell) = \max_{\lambda \in \Lambda(\ell)} \Re(\lambda), \text{ with } \Lambda(\ell) = \left\{ \lambda \in \mathbb{C} \ : \ \det\left(\lambda I_n - \sum_{i=0}^{h} A_i(\ell) e^{-\lambda \tau_i(\ell)} \right) = 0, \right\}, \tag{1}$$

$I_n \in \mathbb{R}^{n\times n}$ is the identity matrix, and $A_i : \mathbb{S} \mapsto \mathbb{C}^{n\times n}$ and $\tau_i : \mathbb{S} \mapsto \mathbb{R}_{\geq 0}$ are smooth functions for every $i \in \{0, \dots, h\}$.

The spectral abscissa $\ell \mapsto \alpha(\ell)$ is a continuous function almost everywhere smooth. In a set of measure zero, however, it can be non-differentiable and not even Lipschitz continuous. This lack of smoothness properties heavily affects the convergence rate of polynomial (chaos) approximations.

Function $\ell \mapsto \alpha(\ell)$ can be approximated on a polynomial basis $\{p_i(\ell)\}_{i=0}^P$: $\alpha_P(\ell) = \sum_{i=0}^P c_i p_i(\ell) \approx \alpha(\ell)$. If ℓ is seen as the realization of a random variable L, which quantifies the uncertainty on physical quantities, then $\alpha(L)$ is a random variable and $\alpha_P(\ell)$ can be seen as a realization of the polynomial chaos approximation $\alpha_P(L)$.

Polynomial approximations of univariate and bivariate spectral abscissa functions, obtained by Galerkin and collocation approaches, are considered. In the Galerkin approach, the numerical approximation of the univariate coefficients c_i is achieved by extended (or composite) Trapezoidal and Simpson's rules or by Gauss and Clenshaw-Curtis quadrature rules. In the bivariate case, the coefficients are approximated by tensorial and non-tensorial Clenshaw-Curtis cubature rules. The collocation approach interpolates the function on Chebyshev points in the univariate case, while for the bivariate case the interpolant nodes are tensor-product Chebyshev grid and Padua points.

The smoothness properties of the spectral abscissa function highly influence the convergence rates of its polynomial (chaos) approximation, leading to a classification according to three possible scenarios. If the spectral abscissa behaves smoothly, then the polynomial approximation converges, as theoretically expected, faster than $\mathcal{O}(P^{-k})$ for all $k \in \mathbb{N}$, while if it is non-differentiable or not even Lipschitz continuous the convergence is at best $\mathcal{O}(P^{-1})$. These two latter cases often occur in control problems, as will be highlighted by the example of the oscillator with feedback delay.

Further information can be found on the preprint [1] and on the tutorial of the experiments [2].

[1] Fenzi L., Michiels W., Polynomial (chaos) approximation of maximum eigenvalue functions: efficiency and limitations, *arXiv*:1804.03881, 2018.

[2] Fenzi L., Michiels W., Experiments on polynomial (chaos) approximation of maximum eigenvalue functions: Tutorial, *Technical report* TW 688, *Department of Computer Science, KU Leuven*, 2018.