Finite-element based observer design for nonlinear systems with delayed measurements

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Abstract: An observer for nonlinear systems with delayed measurements is proposed. The design method is based on computation of a certain manifold defined as a solution of a partial differential equation. This equation is solved using finite-element method. Conditions for existence of a solution are given. Viability of the proposed approach is demonstrated by an example.

Keywords: Delayed-output system, invariant manifold, finite element method.

1. INTRODUCTION

Use of state feedback is the norm in the modern control theory and applications. However, availability of all states of the controlled system is not usual. Rather, in most cases, the values of the unmeasurable states have to be estimated using so-called observer. In the linear case, this task was solved by Luenberger. To estimate the states of a nonlinear system, various methods were developed. First, one can use a linear robust observer where the nonlinearity is treated as an uncertainty. Another approach uses the so-called high gain observers, see e.g. Khalil (2001). A drawback of this method is a rather strong sensitivity to noise.

An alternative approach was proposed by Kazantsis and Kravaris in Kazantzis and Kravaris (1998). The main idea was to find a nonlinear counterpart of the Sylvester equation proposed by Luenberger for the linear case. The solution of this equation, which is a partial differential equation (PDE) in the nonlinear case, is a diffeomorphism between the observer and the observed system.

Time delays occur in many practical applications. Hence the need for reconstruction of the state values for such systems. As the number of results is huge, only some of them can be mentioned here. In many cases, the results are obtained using the Lyapunov-Krasovskii or Razumikhin functionals in combination with linear matrix inequalities. As an example of such papers, see Mahmoud (2011). Paper Ghanes et al. (2016) presents an observer for a nonlinear time-delay system with unknown parameters. State reconstruction in presence of quantized measurements is investigated in Rehák (2017); analysis of the impact of imprecisely known value of the delay is presented there as well. An observer for polynomial systems is described in Rehák (2015).

Observers for systems with delayed output have been studied for a long time. Cacace et al. (2010) presents an observer for a nonlinear system with output measurements. This design method uses exact feedback linearization as the main tool. The time delay might be time-varying. An observer for nonlinear systems with delayed output is presented in Germani et al. (2002): Here, a cascade observer is proposed and convergence of this scheme is proved. The same idea is used also in Kazantzis and Wright (2005). This idea was generalized for systems with multiple delays in Cacace et al. (2014). A predictor for systems with delayed measurements is proposed e.g. in Khosravian et al. (2015). Borri et al. (2017) shows an application of observers in biology (to the model of an artificial pancreas). Recently, an observer designed via the Immersion & Invariance Principle was proposed in Murguia et al. (2016).

As a method to obtain the solution describing the diffeomorphism between the observer and the observed system, Kazantzis and Kravaris (1998) proposes a method based on the Taylor expansions. However, to prove existence of the approximation of the solution, the Lyapunov auxiliary theorem was used. This theorem, in turn, has rather restrictive assumptions - the observed system must be either exponentially stable around the equilibrium or all eigenvalues of its linearization around the origin must have positive real parts. This drawback was removed in the paper Sakamoto et al. (2014) where the diffeomorphism was found using an iterative method, originally proposed for computation of stable, center-stable etc. manifolds Sakamoto and Rehák (2011). This method is based on a successive solution of ordinary differential equations and gained promising results in controlling practical systems, see Tran et al. (2017). The aforementioned equation is related to the PDE that arises in the nonlinear output regulation problem. This PDE was numerically solved using the finite-element method (FEM) in Rehák and Čelikovský (2008); Rehák et al. (2009), further details concerning its solution were presented in Rehák (2011). These results are a base for FEM-based computing of the diffeomorphism first introduced in Kazantzis and Wright (2005).

The contribution of the paper can be summarized into the following points:

- To provide a method for the nonlinear observer design that is easy to implement and has guaranteed convergence on a predefined set,
- To prove existence of the nonlinear observer under less restrictive conditions than in the original papers Kazantzis and Kravaris (1998); Kazantzis and Wright (2005),
- To give the reader a self-contained tutorial for implementation of the presented method.

Notation:

(1) The norm $\|.\|$ is the quadratic norm; in case of matrices, $\|A\|$ is the square root of the maximal eigenvalue of the matrix A^TA.
(2) If $f : [-\tau, \infty) \to R$ is a continuous function then $\|f\|_\infty = \sup\{|f(t)|, t \in [-\tau, 0]\}$ (the quantity $\tau > 0$ will be specified in the sequel).
(3) The time argument is often omitted: $f = f(t)$; the time delay is written in the subscript: $f_\tau = f_\tau(t) = f(t - \tau)$.
(4) The symbol x may represent either a function $x : R \to R^n$ which is a solution of (1) or a vector from R^n. The meaning will be clear from the context.

2. PROBLEM SETTING

The problem setting is based on the papers Kazantzis and Kravaris (1998); Kazantzis and Wright (2005). Hence, the presentation is kept rather short without dealing with details.

The plant to be observed is described by the equation

$$\dot{x} = F(x),\ y = Cx_\tau, \tag{1}$$

where $F : R^n \to R^n$ is a sufficiently smooth function with $F(0) = 0$. Since we will deal with the linear terms in the Taylor polynomial of F in a different way than with the remaining part, it is useful to introduce the matrix $A \in R^{n\times n}$ and a function $f : R^n \to R^n$ vanishing at the origin together with its first derivatives so that for all $x \in R^n$ holds $F(x) = Ax + f(x)$. Then the equation (1) can be reformulated as

$$\dot{x} = Ax + f(x),\ y_\tau = Cx_\tau,\ x(0) = x_0 \tag{2}$$

where $x(t) \in R^n$, $A \in R^{n\times n}$, $C \in R^{1\times n}$.

These assumptions will be crucial in the subsequent text:
Assumptions:

(1) The pair (C, A) is observable.
(2) For any initial condition $x_0 \in R^n$, a unique solution $x : R \to R^n$, $x(0) = x_0$ exists.
(3) The time delay in the output equation $\tau \geq 0$ is assumed to be constant and known.

Remark 2.1. A more general form of the output equation is $y = h(x_\tau)$ for a differentiable function h. However, for the sake simplicity, we assume the output equation is linear. Results similar to those presented in this paper could be derived for the more general function h using the same procedure.

The observer of the system (2) is described by the equation

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L(\hat{x})(y_\tau - \hat{y}_\tau),\ \hat{y} = C\hat{x} \tag{3}$$

where the continuous function $L : R^n \to R^{n\times 1}$ must be found so that the observation error $e(t) = x(t) - \hat{x}(t)$ converges to zero for $t \to \infty$.

Consider a matrix $\tilde{A}$ with all eigenvalues in the open left half-plane. Now we can define the following auxiliary n-dimensional system by

$$\dot{z} = \tilde{A}z + bCx_\tau. \tag{4}$$

Let the mapping $\mathcal{T} : R^n \to R^n$ be defined as $\mathcal{T}(x_0) = x(-\tau)$ with x being solution of (2) with initial condition $x(0) = x_0$. Note that this mapping is well defined thanks to uniqueness of the solution of the differential equation (2). Consider the PDE

$$\frac{\partial \Phi}{\partial x}(Ax + f(x)) = \tilde{A}\Phi(x) + bC\mathcal{T}(x). \tag{5}$$

Let the function $\Phi : R^n \to R^n$ given by the equation (5) be diffeomorphism defined as follows:

$$z = \Phi(x) \tag{6}$$

Let also the vector $b \in R^n$ be such that the pair $(b, \tilde{A})$ is observable.

Proposition 2.2. If this condition is satisfied, then the observer (3) defined with

$$L = \Big(\frac{\partial \Phi}{\partial x}\Big)^{-1} b \tag{7}$$

guarantees convergence of the error to zero.

Proof: Define $\hat{z} = \Phi(\hat{x})$. Then, using (6), one has

$$\begin{aligned}
&\frac{d}{dt}(z - \hat{z})\\
=&\frac{d}{dt}(\Phi(x) - \Phi(\hat{x}))\\
=&(\frac{\partial}{\partial x}\Phi(x))\dot{x} - (\frac{\partial}{\partial x}\Phi(\hat{x}))\dot{\hat{x}}\\
=&\frac{\partial}{\partial x}\Phi(x)(A\Phi(x) + f(x))\\
&-\frac{\partial}{\partial x}\Phi(\hat{x})\Big(A\Phi(\hat{x}) + f(\hat{x}) - (\frac{\partial \Phi}{\partial x}(\hat{x}))^{-1}b(Cx_\tau - C\hat{x}_\tau)\Big)\\
=&\frac{\partial \Phi}{\partial x}(x)(A\Phi(x) + f(x))\\
&-\frac{\partial \Phi}{\partial x}(\hat{x})(A\Phi(\hat{x}) + f(\hat{x})) - bCx_\tau + bC\hat{x}_\tau\\
=&\tilde{A}\Phi(x) + bCx_\tau - \tilde{A}\Phi(\hat{x}) + bC\hat{x}_\tau + bCx_\tau - bC\hat{x}_\tau\\
=&\tilde{A}(z - \hat{z}).
\end{aligned} \tag{8}$$

As $\tilde{A}$ is Hurwitz and Φ is a diffeomorphism, convergence of $e = x - \hat{x}$ to zero for $t \to \infty$ is guaranteed.

3. DIFFEOMORPHISM Φ

The mapping Φ can be found analytically in the linear case. As the result of this case is important even for the nonlinear observer, it is handled here separately. Note that, for linear system (1), $\mathcal{T}(x) = e^{-A\tau}x$ for every x. The observed system and the observer are given in the linear case by the following equation:

$$\dot{x} = Ax,\ \dot{\hat{x}} = A\hat{x} + bC(x_\tau - \hat{x}_\tau) \tag{9}$$

Then the manifold is defined as

$$z = \bar{\Phi}x,\ \bar{\Phi} \in R^{n\times n} \tag{10}$$

where the equation (5) attains the form

$$\bar{\Phi}A = \tilde{A}\bar{\Phi} + bCe^{-A\tau}. \tag{11}$$

The observer is then in the form

$$\dot{\hat{x}} = A\hat{x} + \bar{\Phi}^{-1}b(y_\tau - \hat{y}_\tau) \tag{12}$$

Equation (11) is a Sylvester equation, hence the solution exists if and only if the matrices A and $\tilde{A}$ have no common eigenvalues. This can be easily achieved since the matrix $\tilde{A}$ is a design parameter.

Let us focus attention on the nonlinear case. Assume the matrix $\bar{\Phi}$ satisfies (11). Let us decompose the (unknown) mapping Φ into the linear part $\bar{\Phi}$ and the remaining higher-order terms as follows:

$$\Phi(x) = \bar{\Phi} + \phi(x). \tag{13}$$

The function f is continuously differentiable such that $f(0) = 0$, $\frac{\partial f}{\partial x}(0) = 0$. Using this notation, equation (5) can be rewritten as

$$(\bar{\Phi} + \frac{\partial \phi}{\partial x})(Ax + f(x)) = \tilde{A}(\bar{\Phi}x + \phi(x)) + bC\mathcal{T}(x). \tag{14}$$

Let x be a solution of the system (1) such that $x(0) = x_0$. Then

$$\mathcal{T}(x_0) = x_\tau = e^{-A\tau}x_0 - \int_{t-\tau}^{t} e^{A(t-s)} f(x(s))ds. \tag{15}$$

Using (11) one can rewrite (14) into

$$\begin{aligned}\frac{\partial \phi}{\partial x}(Ax + f(x)) =& \tilde{A}\phi(x) - \bar{\Phi}f(x) \\ &+ bC\Big(\mathcal{T}(x) - e^{-A\tau}x\Big).\end{aligned} \tag{16}$$

Finding a solution the equation (16) involves two important issues: first, finding the mapping $\mathcal{T}$. The following one is solving the equation (16) with this function $\mathcal{T}$.

4. COMPUTATION OF THE FUNCTION $\mathcal{T}$

The approximation of the function $\mathcal{T}$ is constructed using a sequence of functions $\xi_m,\ m \in N$ defined in the sequel.

Definition of the mapping $\mathcal{T}$ implies: $\mathcal{T}(x(t)) = \tilde{x}(t-\tau)$ where the function $\tilde{x}$ obeys the equation

$$\dot{\tilde{x}} = A\tilde{x} + f(\tilde{x}),\ \tilde{x}(0) = x(t) \tag{17}$$

which can be reformulated as

$$\dot{\tilde{x}}(t') = e^{-At'}\tilde{x}(0) - \int_{t}^{t-\tau} e^{-A(t'-s)} f(\tilde{x}(s))ds. \tag{18}$$

Equation (18) allows to define an iterative formula for computation of the mapping $\mathcal{T}$ as follows: let $x \in R^n$. Consider first the following function sequence:

(1) Define the function $\xi_0(x,t)$ as $\xi_0(x,t) = e^{-At}x,\ t \in (-\tau, 0)$.
(2) Let $\xi_m(x,t)$ be defined. Then

$$\xi_{m+1}(x,t) = e^{-At}x + \int_{t-\tau}^{t} e^{A(t-s)} f(\xi_m(x,s))ds. \tag{19}$$

Denote $\alpha = \sup\{\|e^{At}\|, t \in [-\tau, 0]\}$.

Assumption: there exists a positive constant $\varkappa$ so that for every $x_1, x_2 \in R^n$ holds

$$\|f(x_1) - f(x_2)\| \leq \varkappa\|x_1 - x_2\|. \tag{20}$$

Theorem 4.1. Let the sequence of functions $\xi_m(x,t)$ be defined as above and let the condition $\tau\varkappa\alpha < 1$ is satisfied. Then for every $x \in R^n$, the sequence of functions $\xi_m(x,.)$ converges uniformly in t on the interval $(-\tau, 0)$.

Proof: Consider the difference $\|\xi_{m+1}(x,t) - \xi_m(x,t)\|$ for a fixed $x \in R^n$ and fixed arbitrary $t \in [-\tau, 0]$. Then, using (20) one obtains

$$\begin{aligned}&\|\xi_{m+1}(x,t) - \xi_m(x,t)\| \\ =&\| \int_{t-\tau}^{t} e^{A(t-s)}\Big(f(\xi_m(x,s)) - f(\xi_{m-1}(x,s))\Big)ds\| \\ \leq& \int_{t-\tau}^{t} \|e^{A(t-s)}\|\varkappa\|\xi_m(x,.) - \xi_{m-1}(x,.)\|_\infty \\ \leq& \tau\varkappa\alpha\|\xi_m(x,.) - \xi_{m-1}(x,.)\|_\infty.\end{aligned}$$

As this holds for all $t \in [-\tau, 0]$, we obtain

$$\|\xi_{m+1}(x,.) - \xi_m(x,.)\|_\infty \leq \tau\varkappa\alpha\|\xi_m(x,.) - \xi_{m-1}(x,.)\|_\infty.$$

If $\tau\varkappa\alpha < 1$ then there exists a continuous function $\xi(x,.)$ and $\xi_m(x,.) \to \xi(x,.)$ uniformly in the second argument. $\square$

Remark 4.2. Note that uniform convergence of the above sequence with respect to x was not proved.

Let $\xi(x,t) = \lim_{m\to\infty} \xi_m(x,t)$. Theorem 4.1 guarantees continuity of the function $\xi(x,t)$ in the second argument.

The function $\mathcal{T}$ is defined by

$$\mathcal{T}(x) = \xi(x, -\tau) - e^{-A\tau}x. \tag{21}$$

Remark 4.3. It is easy to prove that if f is such that $\lim_{\|x\|\to 0} \frac{\|f(x)\|}{\|x\|^k}$ is finite (that means, f is an $O(\|x\|^k)$ function), then $\xi_k - e^{-A\tau}$ (regarded as functions of x with parameter t) have also this property for every t.

To obtain a precise value of the function $\mathcal{T}$, values of the function $\xi(x,t)$ need to be evaluated for all $x \in R^n$. Since this is not possible, one has to choose a bounded domain $\Omega \subset R^n$ such that $0 \in \Omega$ so that all trajectories of the system lie in this domain. Then, one selects a finite set $\Omega_f \subset \Omega$ so that points of Ω_f are "well distributed" within Ω. Then, one computes the value of the function $\mathcal{T}$ for all points from the set Ω_f. Then, an interpolation of the set $\mathcal{T}(\Omega_f)$ is used as the approximation of the function $\mathcal{T}$. Two points are to be emphasized:

- To obtain a precise value $\mathcal{T}(x)$, an infinite number of iterations is necessary. Hence one has to take a sufficiently large number of iterations as the approximation of this value.
- Choice of the domain Ω as well as the finite set Ω_f depends on the specific example. No detailed hint can be given, rather an expertise of the algorithm's behavior with a couple of trials and errors would lead to a desired result.

5. SOLUTION OF THE EQUATION (16)

Solvability of the equation (16) is the issue solved in this section. Note the equation (5) is linear PDE. Nevertheless, it is rather non-standard: it is a first-order equation, however, it does not correspond to first-order PDEs met in physics, e.g. in form of conservation laws. (see also Sakamoto et al. (2014); a similar equation was solved in Rehák (2011)).

First, denote

$$\beta(x) = x^T A^T + f^T(x). \tag{22}$$

Let $\phi = (\phi_1, \ldots, \phi_n)^T$. Then one can write every element of (16) in form

$$\beta(x)\nabla\phi_i(x) - \sum_{j=1}^{n} \tilde{A}_{ij}\phi_j(x) = -\bar{\Phi}_i f(x) - \mathcal{T}_i(x) \qquad (23)$$

where $\bar{\Phi}_i$ denotes the ith row of the matrix $\bar{\Phi}$ and $\mathcal{T}_i(x)$ stands for the ith element of the vector $\mathcal{T}(x)$.

For simplicity, assume first the matrix $\tilde{A}$ is diagonal: $\tilde{A}_{ii} = \mathrm{diag}(a_1, \ldots, a_n)$. Then one can write for the function $\phi_i, i \in \{1, \ldots, n\}$:

$$\beta(x)\nabla\phi_i(x) - a_i\phi_i(x) = -\bar{\Phi}_i f(x) - \mathcal{T}_i(x) \qquad (24)$$

so that all elements of the vector function ϕ are computed independently.

Conditions for solvability of the equation (24) are summarized in Lemma 1.6 in Roos et al. (1996). Details can also be found in Rehák (2011), here as Lemma II.1. For the reader's convenience, this lemma is repeated here.

Lemma 5.1. Let $\Omega \subset R^n$ be a domain with Lipschitz boundary, $0 \in \Omega$, let $n(x)$ be the outward normal vector at the point $x \in \partial\Omega$. Denote by Γ^- the following set: $\Gamma^- = \{x \in \partial\Omega | n(x).\beta(x) < 0\}$. Further assume

$$a_i - \frac{1}{2}\mathrm{div}\beta(x) > \omega \qquad (25)$$

for some $\omega > 0$. Then the ith equation in (24) is solvable with boundary conditions $\phi_i(x) = 0, x \in \Gamma^-$.

In the case the matrix $\tilde{A}$ is not diagonal but has real eigenvalues, it can be transformed into the Jordan canonical form using a similarity transformation: $\tilde{A} = \Gamma^{-1}J\Gamma$. As shown in Rehák (2011), this transformation changes equation (16) into

$$\beta(\bar{x})\nabla_{\bar{x}}\phi_i(\bar{x}) - J\phi(\bar{x}) = \text{right hand side} \qquad (26)$$

where the operator $\nabla_{\bar{x}}$ denotes the ∇-operator with derivatives with respect to the new coordinates $\bar{x}$, the right-hand side does not depend on the function ϕ. Assume the matrix J can be decomposed into M blocks with dimensions $n_1, \ldots, n_M$. Then the equations containing derivatives of the functions $\phi_{n_1}, \ldots, \phi_{n_M}$ have the same form as the equations (24), hence they can be solved as described in the above case of diagonal matrix $\tilde{A}$. In the next step, the functions $\phi_{n_1-1}, \ldots, \phi_{n_M-1}$ are found by solving the following equation:

$$\beta(x)\nabla\phi_{n_i-1}(x) - a_i\phi_{n_i-1}(x) - \phi_{n_i} = -\bar{\Phi}_{n_i-1}f(x) - \mathcal{T}_{n_i-1}(x). \qquad (27)$$

Thus, the function ϕ_{n_i-1} can be computed with knowledge of function ϕ_{n_i}. In the next step, the function ϕ_{n_i-2} is found using the same reasoning.

On the other hand, if the matrix $\tilde{A}$ has complex eigenvalues, Theorem 5.1 cannot be used as this theorem deals only with scalar equations with real coefficients. On the other hand, the matrix $\tilde{A}$ is a design parameter, thus it can be chosen so that its eigenvalues are real.

Lemma 5.1 constitutes a basis for application of the Finite Element Method for the solution of (24). There exists a large number of commercial or free software to solve this problem.

Let us formulate solvability conditions in terms of matrices A, $\tilde{A}$ and the function f:

Lemma 5.2. Let

$$a_i - \mathrm{Trace}A > 0. \qquad (28)$$

Then there exists a neighborhood of the origin U so that the condition (25) is satisfied in U.

Proof: First, note that $\mathrm{div}\beta(x) = \mathrm{Trace}A + \mathrm{div}f(x)$. Let $\omega_i = a_i - \mathrm{Trace}A$. As the derivatives of the smooth function f vanish at the origin there exists a neighborhood of the origin U so that $\|f(x)\| < \omega_i$ for all $x \in U$, hence (25) holds. □

The following theorem summarizes the main result of the paper.

Theorem 5.3. Let (28) hold for all $i = 1, \ldots, n$. Suppose $\bar{\Phi}$ is the solution of (11) and ϕ solves (16). Suppose also the matrix $\tilde{A}$ has real eigenvalues. Then the observer (3) with

$$L(\hat{x}) = \left(\bar{\Phi} + \frac{\partial\phi}{\partial x}(\hat{x})\right)^{-1} b \qquad (29)$$

guarantees $\lim_{t\to\infty}\|e(t)\| = 0$.

Proof: Follows from the Lemma 5.2, Theorem 5.1 and Proposition 2.2. □

Remark 5.4. Due to the observability assumption for the pair (C, A), the matrix $\bar{\Phi}$ is nonsingular. As the mapping ϕ contains terms of order higher than 1, the inversion in the formula (29) is well defined on a neighborhood of the origin.

Remark 5.5. If the delay is present, the mapping $\mathcal{T}$ must be found. This is a nontrivial task since in the nonlinear case, this mapping cannot be determined analytically. Usually, a numerical approximation must be used. First, a finite set of initial conditions $\Delta \subset R^n$ is chosen. For every $\xi \in \Delta$, the solution of (2) on the interval $[-\tau, 0]$ with terminal condition $x(0) = \xi$ is found. Clearly, $\mathcal{T}(\xi) = x(-\tau) - e^{-A\tau}\xi$. Hence, one takes approximation of $\mathcal{T}$ computed from the finite number of values $\mathcal{T}(\xi)$.

Remark 5.6. Kazantzis and Kravaris (1998) presents solvability conditions of the linear equation (11) obtained using the Lyapunov's auxiliary theorem. The conditions derived this way are rather restrictive as the observed system (2) must have all eigenvalues with negative real part or all eigenvalues must have positive real part. Theorem 5.1 implies solvability of the equation (11) under the weaker condition (28).

Remark 5.7. The boundary condition on Γ^- (if necessary) brings some unwanted error in. In the ideal case, the equation is solved on the whole space R^n so that this condition is not needed. However, numerical experiments show that significant influence of these boundary conditions is restricted only on a narrow region around the border of Ω.

6. EXAMPLE

The example system is given by the equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_1^3 \\ y &= x_{1,\tau} \end{aligned}$$

where the observation delay is $\tau = 0.15s$. In this case, $C = (1, 0)$.

We choose

$$\tilde{A} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 0.18 \\ 0.18 \end{pmatrix}$$

Then, for the solution of equation (11) holds

$$\bar{\Phi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \bar{\Phi} + \begin{pmatrix} 0.18 & 0 \\ 0.18 & 0 \end{pmatrix} \exp\Big(\begin{pmatrix} 0 & -0.25 \\ 0.25 & 0 \end{pmatrix} \Big) \tag{30}$$

and solvability conditions are satisfied. Thus, equation (30) has the solution

$$\bar{\Phi} = \begin{pmatrix} 0.1675 & -0.0125 \\ 0.103 & 0.0260 \end{pmatrix}.$$

The iterations were carried out for values of x in the rectangular grid

$$\mathcal{G} = \{(x_i, y_j) | x_i = -2+0.1i, y_j = -2+0.1j, i, j = 0, \ldots, 40\}.$$

Practical simulations reveal this convergence is quite fast as shown in Figure 1. This figure depicts iterations $\xi_k(x, -0.15)$ for $x = (-1.5, 1)$. One can see that after 5 iterations, the values in the iterations does not change. A similar picture could be done for all initial values in the set $\mathcal{G}$.

Interpolation of the results yields

$$\begin{aligned}\Big(\mathcal{T}(x)\Big)_1 =& -0.0110x_1^3 + 0.0016x_1^2x_2 - 0.0002x_1x_2^2 \\ \Big(\mathcal{T}(x)\Big)_2 =& 0.142x_1^3 - 0.0314x_1^2x_2 + 0.004x_1x_2^2 - 0.0003x_2^3.\end{aligned}$$

Solution of the equation (16) on the set $\Omega = \{x \in R^n | \|x\| \leq 2\}$ and subsequent interpolation by a third order polynomial yields

$$\begin{aligned}\phi_1(x_1, x_2) =& 0.011x_1^3 - 0.0013x_1^2x_2 + 0.0022x_1x_2^2 \\ & - 0.0024x_2^3 \\ \phi_2(x_1, x_2) =& -0.0088x_1^3 + 0.0154x_1^2x_2 - 0.0239x_1x_2^2 \\ & + 0.0246x_2^3.\end{aligned}$$

First order terms are missing thanks to the definition of the function ϕ. Note the function ϕ contains no second order terms due to absence of second order terms in the functions f and $\mathcal{T}$. The observer is constructed using the function ϕ as shown in equation (12). The PDE (16) was numerically solved using the finite-element method. The software package Comsol Multiphysics was used. Figure 2 shows the function ϕ_1 computed by this numerical software.

The results of simulations are in figures 3 and 4. Fig. 3 shows the state x_2 (dashed line) and its estimate (solid line). The observation error $e_2 = x_2 - \hat{x}_2$ is depicted in Fig. 4.

7. COMPARISON WITH THE METHOD BASED ON TAYLOR EXPANSIONS

Paper Kazantzis and Wright (2005) (and, in the delay-free case, Kazantzis and Kravaris (1998)) use expansions to Taylor polynomials to find the solution of the equation (5). The right-hand side as well as the function f are approximated by their Taylor polynomials, the solution is sought also in form of a Taylor polynomial. This is probably the most widely used method for solution of PDEs of this type. While this method is easy to explain,

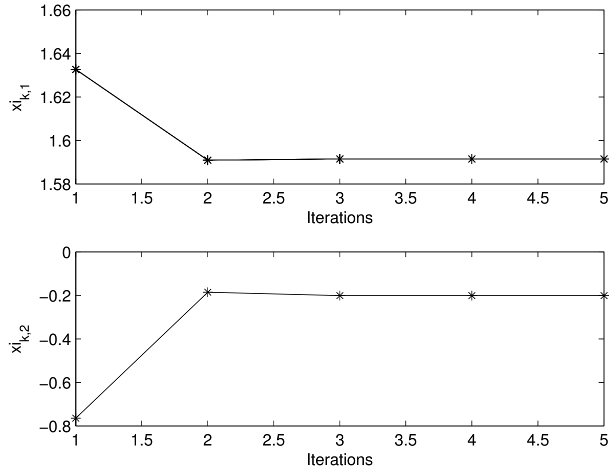

Fig. 1. Values $\xi_k(x, -\tau)$, $k = 1, \ldots, 5$

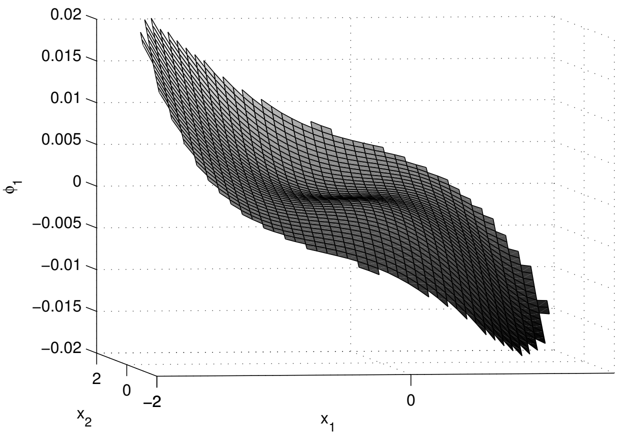

Fig. 2. Function ϕ_1

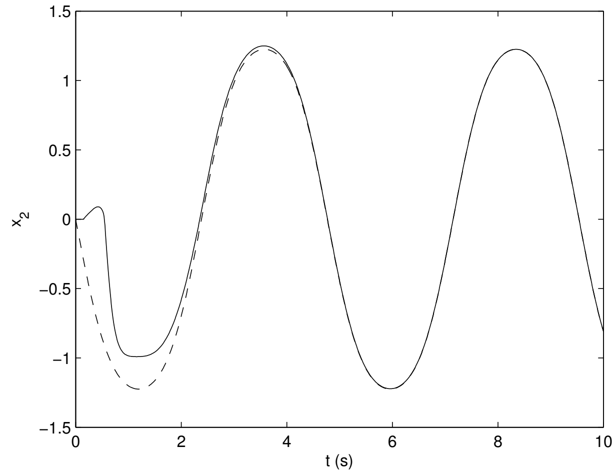

Fig. 3. State x_2 and its estimate

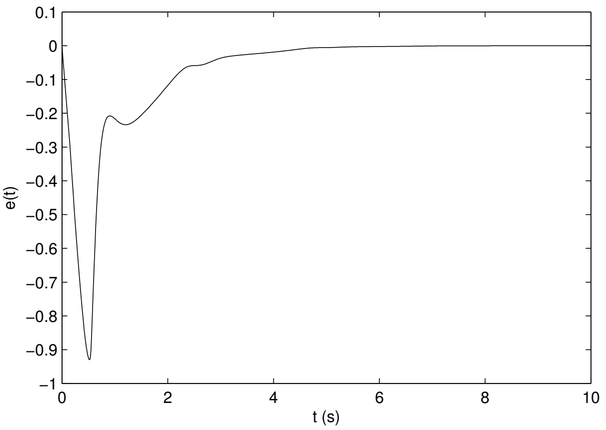

Fig. 4. Observation error in state x_2

it has several drawbacks. First, the result is only local; it is not easy to determine how well the Taylor polynomials approximate the solution at points not equal to the origin. This uncertainty increases with increasing distance from the origin. In contrast, the method presented here gives results whose precision on an a-priori given set (the set Ω) can be easily verified. Moreover, computation of the Taylor polynomials requires lengthy calculations, they are difficult to obtain without help of a symbolic software.

Note also the condition $\tau\varkappa\alpha < 1$ in Theorem 4.1 restricts the maximum allowable time delay. For larger delays, one might divide this delay into ν shorter segments: $0 > \tau_1 > \cdots > \tau_{\nu-1} > \tau_\nu = -\tau$ such that this condition is satisfied on every interval (τ_i, τ_{i-1}). Then, it is possible to compute $\mathcal{T}^1$ on the interval $(0, \tau_1)$ using the procedure described in Section 4. After that, one computes $\mathcal{T}^2(x)$ using the value $\mathcal{T}^1(x)$ etc. This stepwise computation can be in some sense be regarded as a counterpart of the cascading observers described by e.g. Cacace et al. (2014).

8. CONCLUSIONS

A numerical method for design of a nonlinear observer with delayed measurements was presented. The method is based on solution of a partial differential equation using finite-element method. Conditions of convergence of the method were derived. Viability of the method was illustrated by an example.

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