Stability analysis for systems with time-varying delay via orthogonal-polynomial-based integral inequality

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Abstract: This paper proposes an integral inequality related to the state vector for systems with time-varying delay and exploits component vectors of the proposed inequality for constructing a Lyapunov-Krasovskii functional. The proposed inequality is based on orthogonal-polynomial-based integral inequality. The component vectors of the proposed inequality have the relation in terms of time-varying delay with those of the orthogonal-polynomial-based integral inequality. Also, the time-derivative of the component vectors of the proposed inequality are represented by those of the orthogonal-polynomial-based integral inequality. The Lyapunov-Krasovskii functional is constructed by utilizing the component vectors of the proposed inequality and the orthogonal-polynomial-based integral inequality. Based on the the Lyapunov-Krasovskii functional, a stability criterion is derived in terms of linear matrix inequalities. Simulation results show that the proposed criterion is less conservative than the criteria in the literature.

Keywords: stability anlaysis, Lyapunov-Krasovskii functionals, time-delay systems, linear matrix inequality.

1. INTRODUCTION

In the past decades, stability analysis for time-delay systems has widely studied by academic and industrial communities, because time-delay is an inevitable phenomenon in many fields such as chemistry, biology, and mechanical engineering (Gu et al., 2003). In stability analysis, an important issue is to obtain maximum delay bounds guaranteeing the asymptomatic stability of the time-delay systems. Therefore, many stability criteria formulated in terms of linear matrix inequality (LMI) conditions have proposed by various Lyapunov-Krasovskii functionals (LKFs) and integral inequalities (Seuret and Gouaisbaut, 2013; Kwon et al., 2014; Park et al., 2015; Zeng et al., 2015; Zhang et al., 2016; Lee et al., 2017a,b; Zhi et al., 2017; Zhang et al., 2017).

There are two approaches to obtain less conservative criteria. One is to develop lower bound lemmas of integral terms in LKFs such as Jensen's inequality(Jensen, 1906), Wirtinger-based inequality(Seuret and Gouaisbaut, 2013), auxiliary function-based integral inequality(Park et al., 2015), free-matrix-based integral inequality(Zeng et al., 2015), improved free-matrix-based integral inequality(Zhi et al., 2017), Bessel-Legendre inequality(Seuret and Gouaisbaut, 2014), and polynomial-based integral inequality(Lee et al., 2017b). Recently, (Lee et al., 2017a) proposed orthogonal-polynomial-based integral inequality which provides less conservative results than Bessel-Legendre inequality and less computational complexity than polynomial-based integral inequality. In (Lee et al., 2017a), the upper bounds of $-\int_{t-h(t)}^{t}\dot{x}^T(r)R\dot{x}(r)dr$ and $-\int_{t-h}^{t-h(t)}\dot{x}^T(r)R\dot{x}(r)dr$ were provided. The other is to choose appropriate LKFs with the augmented vectors to provide more freedom for checking the feasibility of the LMI conditions (Kim, 2016; Zhang et al., 2017). In (Kim, 2016), augmented vectors with integral vectors are used in a quadratic term and integral terms of the LKF. In (Zhang et al., 2017), augmented vectors involved in current and delayed states, a time-derivative state, and the integral vectors are used in integral terms of the LKF. The time-derivative of integral terms with the augmented vectors of the LKFs in (Kim, 2016; Zhang et al., 2017) are a convex function with respect to $\dot{h}(t)$, but a quadratic function with respect to $h(t)$. There is still room to obtain a less conservative criterion by proposing an integral inequality related to $-\int_{t-h(t)}^{t}x^T(r)Rx(r)dr$ and $-\int_{t-h}^{t-h(t)}x^T(r)Rx(r)dr$ whose component vectors make the quadratic function to be convex with respect to $h(t)$.

This paper proposes an integral inequality, which is related to $-\int_{t-h(t)}^{t}x^T(r)Rx(r)dr$ and $-\int_{t-h}^{t-h(t)}x^T(r)Rx(r)dr$, for systems with time-varying delay and exploits component vectors of the proposed inequality for constructing a LKF. The proposed inequality is based on orthogonal-polynomial-based integral inequality. The component vectors of the proposed inequality have the relation in terms of $h(t)$ with those of the orthogonal-polynomial-based integral inequality. Also, the time-derivative of the component vectors of the proposed inequality are represented by those of the orthogonal-polynomial-based integral inequality. The LKF is constructed by utilizing the component vectors of the proposed inequality and the orthogonal-polynomial-based integral inequality. Based on the the LKF, a stability criterion derived in terms of LMIs is convex with respect to $h(t)$ and $\dot{h}(t)$, respectively. Simulation results show that the proposed criterion is less conservative than the criteria in the literature.

Notations. Throughout this paper, the $\mathbb{R}^n$ is n-dimensional vectors. The superscripts '-1' and 'T' denote the inverse and transpose of a matrix. $P > 0$ means the matrix, P, is symmetric and positive definite. Symmetric terms in a matrix are denoted by $*$. $diag\{\dots\}$ stands for a block diagonal matrix. $col\{x_1, x_2, \dots, x_n\}$ means $[x_1^T, x_2^T, \dots, x_n^T]^T$. He $\{Z\} = Z + Z^T$.

2. PRELIMINARIES

Consider the following system with a time-varying delay described by

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - h(t)) \\ x(\theta) = \phi(\theta), \quad \theta \in [-h, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\phi(\theta)$ is an initial condition, and $h(t)$ is time-varying delay satisfying

$$0 \le h(t) \le h, \quad -d \le \dot{h}(t) \le d < \infty \tag{2}$$

where h and d are known constants.

An orthogonal polynomial function called as Legendre polynomial function over an interval $[a, b]$ is defined as follows.

Definition 1. (Seuret and Gouaisbaut, 2014) Legendre polynomials over the interval $[a, b]$ can be defined as follows.

$$L_i(r) = \sum_{j=0}^{i} l_j^i \left(\frac{r-a}{b-a} \right)^j \quad \text{for } r \in [a, b], \tag{3}$$

where

$$l_j^i = (-1)^{i+j} \begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix}. \tag{4}$$

The polynomial function satisfies the following properties:

$$1)\ L_k(b) = 1, \quad L_k(a) = (-1)^k, \tag{5}$$

$$2)\ \int_a^b L_k(r) L_l(r) dr = \begin{cases} 0 & \text{if } k \neq l, \\ \frac{b-a}{2k+1} & \text{if } k = l \end{cases}. \tag{6}$$

The following lemmas are used for obtaining a proposed stability criterion.

Lemma 2. (Lee et al., 2017b) For an non-negative integer m, let $x(r) \in \mathbb{R}^n$ be an integrable function: $\{x(r) | r \in [a, b]\}$. Then we have

$$\int_a^b (r-a)^m x(r) dr = m! \mathbb{I}_m(a, b), \tag{7}$$

where

$$\mathbb{I}_m(a.b) = \int_a^b \int_{r_1}^b \cdots \int_{r_m}^b x(r_{m+1}) dr_{m+1} \cdots dr_1. \tag{8}$$

Lemma 3. (Lee et al., 2017a) Let $x(r) \in \mathbb{R}^n$ be a continuous function: $\{x(r) | r \in [a, b]\}$. For a non-negative integer m, a positive integer k, an arbitrary vector $\zeta \in \mathbb{R}^{kn}$, a positive definite matrix R, and a matrix F with appropriate dimensions, the following inequality holds:

$$\begin{aligned} &-\int_a^b \dot{x}^T(r) R \dot{x}(r) dr \\ &\le (b-a)\zeta^T F R_m^{-1} F^T \zeta + \text{He}\{\zeta^T F \mathbb{L}(a, b)\}, \end{aligned} \tag{9}$$

where

$$\mathbb{L}(a, b) = \text{col}\{\mathbb{L}_0(a, b), \dots, \mathbb{L}_m(a, b)\}, \tag{10}$$

$$R_m = \text{diag}\{R, 3R, \dots, (2m+1)R\}, \tag{11}$$

$$\begin{aligned} &\mathbb{L}_i(a, b) \\ &= \begin{cases} x(b) - x(a) & \text{if } i = 0 \\ x(b) - (-1)^i x(a) - \sum_{j=1}^{i} l_j^i \frac{j!}{(b-a)^j} \mathbb{I}_{j-1}(a, b) & \text{for } i \in \mathbb{N} \end{cases}. \end{aligned} \tag{12}$$

Lemma 4. Let $x(r) \in \mathbb{R}^n$ be a continuous function: $\{x(r) | r \in [a, b]\}$. For a non-negative integer m, a positive integer k, an arbitrary vector $\zeta \in \mathbb{R}^{kn}$, a positive definite matrix R, and a matrix F with appropriate dimensions, the following inequality holds:

$$\begin{aligned} &-\int_a^b x^T(r) R x(r) dr \\ &\le (b-a)\zeta^T F R_m^{-1} F^T \zeta + \text{He}\{\zeta^T F \mathbb{M}(a, b)\}, \end{aligned} \tag{13}$$

where

$$\mathbb{M}(a, b) = \text{col}\{\mathbb{M}_0(a, b), \dots, \mathbb{M}_m(a, b)\}, \tag{14}$$

$$R_m = \text{diag}\{R, 3R, \dots, (2m+1)R\}, \tag{15}$$

$$\mathbb{M}_i(a, b) = \sum_{j=0}^{i} l_j^i \frac{j!}{(b-a)^j} \mathbb{I}_j(a, b) \qquad \text{for } i \in \mathbb{N}. \tag{16}$$

Proof. For matrices $Y_i \in \mathbb{R}^{kn \times n}$ $(i = 0, \dots, m)$, let us define

$$F = [Y_0 \dots Y_m], \quad Y = \text{col}\{Y_0, \dots, Y_m\}, \tag{17}$$

$$\bar{R} = \begin{bmatrix} Y R^{-1} Y^T & Y \\ Y^T & R \end{bmatrix}, \quad \mathbb{M}_i(a, b) = \int_a^b L_i(r) x(r) dr, \tag{18}$$

$$\zeta_m = \text{col}\{L_0(r)\zeta, \dots, L_m(r)\zeta\}. \tag{19}$$

Due to the positive definite matrix R, it is clear that $\bar{R} > 0$. Then, the following inequality holds

$$\begin{aligned} &-\int_a^b x^T(r) R x(r) dr \\ &\le \int_a^b \zeta_m^T Y R^{-1} Y^T \zeta_m dr + \text{He}\left\{ \int_a^b \zeta_m^T Y x(r) dr \right\}. \end{aligned} \tag{20}$$

From (5), (6), and , (20) is represented as

$$\begin{aligned} &-\int_a^b x^T(r) R x(r) dr \\ &\le \sum_{i=0}^{m} \left(\frac{b-a}{2i+1} \zeta^T Y_i R^{-1} Y_i^T \zeta + \text{He}\left\{ \zeta^T Y_i \mathbb{M}_i(a, b) \right\} \right). \end{aligned} \tag{21}$$

Rearranging (21) yields (13). It completes the proof $\square$.

Remark 5. This paper proposes the integral inequality of $-\int_a^b x^T(r) R x(r) dr$ based on the orthogonal-polynomial-based integral inequality (Lee et al., 2017a). The component vectors of the proposed inequality have the following relations with those of the orthogonal-polynomial-based integral inequality: For $i = 0$, the time-derivative $\mathbb{I}_0(a, b)$ of $\mathbb{M}_0(a, b)$ includes $x(b)$ and $x(a)$ related to $\mathbb{L}_0(a, b)$. For $i \ge 1$, the time-derivative $(b-a)^{-i} \mathbb{I}_i(a, b)$ of $\mathbb{M}_i(a, b)$ includes $(b-a)^{-i} \mathbb{I}_{i-1}(a, b)$ and $(b-a)^{-(i+1)} \mathbb{I}_i(a, b)$ related to $\mathbb{L}_i(a, b)$ and $\mathbb{L}_{i+1}(a, b)$, respectively. Also, for $i \ge 0$, the following zero-equalities can be obtained: $(b-a)^{-i} \mathbb{I}_i(a, b) - (b-a)(b-a)^{-(i+1)} \mathbb{I}_i(a, b) = 0$, where $(b-a)^{-i} \mathbb{I}_i(a, b)$ is the component vectors of $\mathbb{M}_i(a, b)$ and $(b-a)^{-(i+1)} \mathbb{I}_i(a, b)$ is the component vectors of $\mathbb{L}_{i+1}(a, b)$.

3. MAIN RESULTS

This section proposes a stability criterion for the time-delay system (1).

Theorem 6. For given scalars h and d, the time-delay system (1) is asymptotically stable if there exist positive definite matrices $P \in \mathbb{R}^{7n \times 7n}$, $Q_1, Q_2 \in \mathbb{R}^{6n \times 6n}$, $R_1, R_2 \in \mathbb{R}^{n \times n}$, and matrices $F_1, F_2 \in \mathbb{R}^{13n\times 3n}$, $F_3, F_4 \in \mathbb{R}^{13n\times 2n}$, $G_i \in \mathbb{R}^{13n\times n}$ $(i = 1, \ldots, 4)$ such that

$$\begin{bmatrix} \Phi_{[h,d]} & \sqrt{h}F_1 & \sqrt{h}F_3 \\ * & -R_{1_m} & 0 \\ * & * & -R_{2_m} \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} \Phi_{[0,d]} & \sqrt{h}F_2 & \sqrt{h}F_4 \\ * & -R_{1_m} & 0 \\ * & * & -R_{2_m} \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} \Phi_{[h,-d]} & \sqrt{h}F_1 & \sqrt{h}F_3 \\ * & -R_{1_m} & 0 \\ * & * & -R_{2_m} \end{bmatrix} < 0, \tag{24}$$

$$\begin{bmatrix} \Phi_{[0,-d]} & \sqrt{h}F_2 & \sqrt{h}F_4 \\ * & -R_{1_m} & 0 \\ * & * & -R_{2_m} \end{bmatrix} < 0, \tag{25}$$

where $\Phi_{[h(t),\dot{h}(t)]} = h\mathscr{D}_0^T R_1 \mathscr{D}_0 + he_1^T R_2 e_1 + \mathscr{D}_{20}^T Q_1 \mathscr{D}_{20}$ $-(1-\dot{h}(t))\mathscr{D}_{21}^T Q_1 \mathscr{D}_{21} + (1-\dot{h}(t))\mathscr{D}_{30}^T Q_2 \mathscr{D}_{30} - \mathscr{D}_{31}^T Q_2 \mathscr{D}_{31}$ $+\text{He}\{\mathscr{D}_1^T P \mathscr{D}_{11} + \mathscr{D}_{22}^T Q_1 \mathscr{D}_{23} + \mathscr{D}_{32}^T Q_2 \mathscr{D}_{33} + F_1 \mathscr{D}_{41} + F_2 \mathscr{D}_{42}$ $+F_3 \mathscr{D}_{51} + F_4 \mathscr{D}_{52} + G_1 \mathscr{D}_6 + G_2 \mathscr{D}_7 + G_3 \mathscr{D}_8 + G_4 \mathscr{D}_9\}$ with

$\mathscr{D}_0 = Ae_1 + A_d e_2$, $\mathscr{D}_1 = \text{col}\{e_1, e_2, e_3, e_8, e_9, e_{10}, e_{11}\}$,
$\mathscr{D}_{11} = \text{col}\{\mathscr{D}_0, (1-\dot{h}(t))e_{12}, e_{13}, e_1 - (1-\dot{h}(t))e_2,$
$(1-\dot{h}(t))e_2 - e_3, e_1 - (1-\dot{h}(t))e_4 - \dot{h}(t)e_6,$
$(1-\dot{h}(t))e_2 - e_5 + \dot{h}(t)e_7\}$,
$\mathscr{D}_{20} = \text{col}\{e_1, \mathscr{D}_0, e_0, e_1, e_2, e_3\}$,
$\mathscr{D}_{21} = \text{col}\{e_2, e_{12}, e_8, e_1, e_2, e_3\}$,
$\mathscr{D}_{22} = \text{col}\{e_8, e_1 - e_2, h(t)e_{10}, h(t)e_1, h(t)e_2, h(t)e_3\}$,
$\mathscr{D}_{23} = \text{col}\{e_0, e_0, e_1, \mathscr{D}_0, (1-\dot{h}(t))e_{12}, e_{13}\}$,
$\mathscr{D}_{30} = \text{col}\{e_2, e_{12}, e_0, e_1, e_2, e_3\}$,
$\mathscr{D}_{31} = \text{col}\{e_3, e_{13}, e_9, e_1, e_2, e_3\}$,
$\mathscr{D}_{32} = \text{col}\{e_9, e_2 - e_3, (h-h(t))e_{11}, (h-h(t))e_1,$
$(h-h(t))e_2, (h-h(t))e_3\}$,
$\mathscr{D}_{33} = \text{col}\{e_0, e_0, (1-\dot{h}(t))e_2, \mathscr{D}_0, (1-\dot{h}(t))e_{12}, e_{13}\}$,
$\mathscr{D}_{41} = \text{col}\{e_1 - e_2, e_1 + e_2 - 2e_4, e_1 - e_2 + 6e_4 - 12e_6\}$,
$\mathscr{D}_{42} = \text{col}\{e_2 - e_3, e_2 + e_3 - 2e_5, e_2 - e_3 + 6e_5 - 12e_7\}$,
$\mathscr{D}_{51} = \text{col}\{e_8, -e_8 + 2e_{10}\}$, $\mathscr{D}_{52} = \text{col}\{e_9, -e_9 + 2e_{11}\}$,
$\mathscr{D}_6 = h(t)e_4 - e_8$, $\mathscr{D}_7 = (h-h(t))e_5 - e_9$,
$\mathscr{D}_8 = h(t)e_6 - e_{10}$, $\mathscr{D}_9 = (h-h(t))e_7 - e_{11}$,
$e_i = [0_{n\times(i-1)n}\ I_n\ 0_{n\times(13-i)n}]$, $i = 1, \ldots, 13$,
$e_0 = [0_{n\times 13n}]$.

Proof. Choose the following LKF $V(t)$ such that

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \tag{26}$$

where

$$\begin{aligned} V_1(t) &= \eta_1^T(t) P \eta_1(t), \\ V_2(t) &= \int_{t-h(t)}^{t} \eta_2^T(t,s) Q_1 \eta_2(t,s) ds \\ &\quad + \int_{t-h}^{t-h(t)} \eta_3^T(t,s) Q_2 \eta_3(t,s) ds, \\ V_3(t) &= \int_{-h}^{0}\int_{t+s}^{t} \dot{x}^T(r) R_1 \dot{x}(r) dr ds, \\ V_4(t) &= \int_{-h}^{0}\int_{t+s}^{t} x^T(r) R_2 x(r) dr ds \end{aligned}$$

with $\eta_1(t) = \text{col}\{\eta_0(t), \int_{t-h(t)}^{t} x(s)ds, \int_{t-h}^{t-h(t)} x(s)ds,$ $\frac{1}{h(t)}\int_{t-h(t)}^{t}\int_s^t x(r)drds, \frac{1}{h-h(t)}\int_{t-h}^{t-h(t))}\int_s^{t-h(t)} x(r)drds\}$,
$\eta_2(t,s) = \text{col}\{x(s), \dot{x}(s), \int_s^t x(r)dr, \eta_0(t)\}$,
$\eta_3(t,s) = \text{col}\{x(s), \dot{x}(s), \int_s^{t-h(t)} x(r)dr, \eta_0(t)\}$,
where $\eta_0(t) = \text{col}\{x(t), x(t-h(t)), x(t-h)\}$.
Then, the time-derivative of $V(t)$ can be derived as

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t), \tag{27}$$

where

$$\begin{aligned} \dot{V}_1(t) &= \text{He}\{\eta_1^T(t) P \dot{\eta}_1(t)\}, \\ \dot{V}_2(t) &= \eta_2^T(t,t) Q_1 \eta_2(t,t) \\ &\quad - (1-\dot{h}(t))\eta_2^T(t,t-h(t)) Q_1 \eta_2(t,t-h(t)) \\ &\quad + (1-\dot{h}(t))\eta_3^T(t,t-h(t)) Q_2 \eta_3(t,t-h(t)) \\ &\quad - \eta_3^T(t,t-h) Q_2 \eta_3(t,t-h) \\ &\quad + \text{He}\Big\{\int_{t-h(t)}^{t} \eta_2^T(t,s) Q_1 \frac{\partial \eta_2(t,s)}{\partial t} ds \\ &\quad + \int_{t-h}^{t-h(t)} \eta_3^T(t,s) Q_2 \frac{\partial \eta_3(t,s)}{\partial t} ds\Big\}, \\ \dot{V}_3(t) &= h\dot{x}^T(t) R_1 \dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds, \\ \dot{V}_4(t) &= hx^T(t) R_2 x(t) - \int_{t-h}^{t} x^T(s) R_2 x(s) ds. \end{aligned}$$

Applying Lemmas 3 and 4 to the integral terms of $\dot{V}_3(t)$ and $\dot{V}_4(t)$ lead to

$$\begin{aligned} &-\int_{t-h}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds \\ &\le h(t)\zeta^T F_1 R_{1m}^{-1} F_1^T \zeta + (h-h(t))\zeta^T F_2 R_{1m}^{-1} F_2^T \zeta \\ &\quad + \text{He}\{\zeta^T F_1 \mathbb{L}_1(t-h(t),t) + \zeta^T F_2 \mathbb{L}_2(t-h,t-h(t))\}, \end{aligned} \tag{28}$$

$$\begin{aligned} &-\int_{t-h}^{t} x^T(s) R_2 x(s) ds \\ &\le h(t)\zeta^T F_3 R_{2m}^{-1} F_3^T \zeta + (h-h(t))\zeta^T F_4 R_{2m}^{-1} F_4^T \zeta \\ &\quad + \text{He}\{\zeta^T F_3 \mathbb{M}_1(t-h(t),t) + \zeta^T F_4 \mathbb{M}_2(t-h,t-h(t))\}, \end{aligned} \tag{29}$$

where $\mathbb{L}_1(t-h(t),t) = \mathscr{D}_{41}\zeta$, $\mathbb{L}_2(t-h,t-h(t)) = \mathscr{D}_{42}\zeta$, $\mathbb{M}_1(t-h(t),t) = \mathscr{D}_{51}\zeta$, and $\mathbb{M}_2(t-h,t-h(t)) = \mathscr{D}_{52}\zeta$ with $\zeta = \text{col}\{x(t), x(t-h(t)), x(t-h), \frac{1}{h(t)}\int_{t-h(t)}^{t} x(s)ds,$ $\frac{1}{h-h(t)}\int_{t-h}^{t-h(t)} x(s)ds, \frac{1}{h^2(t)}\int_{t-h(t)}^{t}\int_s^t x(r)drds,$ $\frac{1}{(h-h(t))^2}\int_{t-h}^{t-h(t)}\int_s^{t-h(t)} x(r)drds, \int_{t-h(t)}^{t} x(s)ds,$ $\int_{t-h}^{t-h(t)} x(s)ds, \frac{1}{h(t)}\int_{t-h(t)}^{t}\int_s^t x(r)drds,$ $\frac{1}{h-h(t)}\int_{t-h}^{t-h(t)}\int_s^{t-h(t)} x(r)drds, \dot{x}(t-h(t)), \dot{x}(t-h)\}$.
From the relations between entries in $\mathbb{L}_i$ and $\mathbb{M}_i$ $(i = 1, 2)$, the following zero-equalities can be obtained:

$$\text{He}\{\zeta^T G_1 [h(t)e_4 - e_8]\zeta\} = 0, \tag{30}$$

$$\text{He}\{\zeta^T G_2 [(h-h(t))e_5 - e_9]\zeta\} = 0, \tag{31}$$

$$\text{He}\{\zeta^T G_3 [h(t)e_6 - e_{10}]\zeta\} = 0, \tag{32}$$

$$\text{He}\{\zeta^T G_4 [(h-h(t))e_7 - e_{11}]\zeta\} = 0. \tag{33}$$

By using integral inequalities (28) and(29) and zero-equalities (30)-(33), the following stability criterion is obtained:

$$\dot{V}(t) \le \zeta^T [\Phi_{[h(t),\dot{h}(t)]} + h(t)\Gamma_1 + (h-h(t))\Gamma_2]\zeta, \tag{34}$$

where $\Gamma_1 = F_1 R_{1m}^{-1} F_1^T + F_3 R_{2m}^{-1} F_3^T$, $\Gamma_2 = F_2 R_{1m}^{-1} F_2^T + F_4 R_{2m}^{-1} F_4^T$.

Table 1. The maximum admissible upper bound h for Example 1

Method	d			
	0.1	0.2	0.5	0.8
Kim (2016)	4.753	3.857	2.429	2.183
Zeng et al. (2015)	4.788	4.060	3.055	2.615
Kwon et al. (2014)	4.811	4.101	3.061	2.612
Zhi et al. (2017)	4.836	4.149	3.157	2.708
Zhang et al. (2017)	4.910	4.216	3.233	2.789
Theorem 6	4.955	4.300	3.380	2.973

Therefore, the time-delay system (1) is asymptotically stable if the following condition holds

$$\Phi_{[h(t),\dot{h}(t)]} + h(t)\Gamma_1 + (h - h(t))\Gamma_2 < 0. \quad (35)$$

Here, $\Phi_{[h(t),\dot{h}(t)]}$ is convex with respect to $h(t) \in [0, h]$ and $\dot{h}(t) \in [-d, d]$, respectively. Therefore, the condition (35) is also convex with respect to $h(t) \in [0, h]$ and $\dot{h}(t) \in [-d, d]$, respectively. Based on Schur complement, the condition (35) is equivalent to (22)-(25). It completes the proof $\square$.

Remark 7. In Theorem 6, an augmented vector ζ of the stability criterion (34) includes single integral vectors and double integral vectors of $\mathbb{M}_1(t - h(t), t)$, $\mathbb{M}_2(t - h, t - h(t))$, $\mathbb{L}_1(t - h(t), t)$ and $\mathbb{L}_2(t - h, t - h(t))$. For the augmented vector ζ, the time-derivative of $V_2(t)$ can be represented as a convex condition with respect to $h(t) \in [0, h]$ and $\dot{h}(t) \in [-d, d]$, respectively. However, for the augmented vector ζ without single integral vectors and double integral vectors of $\mathbb{M}_1(t - h(t), t)$ and $\mathbb{M}_2(t - h, t - h(t))$, the time-derivative of $V_2(t)$ is represented as a convex condition with respect to $\dot{h}(t) \in [-d, d]$, but a quadratic condition with respect to $h(t) \in [0, h]$.

Remark 8. In $V_1(t)$, the single and double integral vectors of $\mathbb{M}_1(t - h(t), t)$ and $\mathbb{M}_2(t - h, t - h(t))$ are included in $\eta_1(t)$. These integral vectors have the following relations with those of $\mathbb{L}_1(t - h(t), t)$ and $\mathbb{L}_2(t - h, t - h(t))$:

$$\frac{d}{dt}\left(\int_{t-h(t)}^{t} x(s)ds\right) = x(t) - (1 - \dot{h}(t))x(t - h(t)),$$

$$\frac{d}{dt}\left(\int_{t-h}^{t-h(t)} x(s)ds\right) = (1 - \dot{h}(t))x(t - h(t)) - x(t - h),$$

$$\frac{d}{dt}\left(\frac{1}{h(t)}\int_{t-h(t)}^{t}\int_{s}^{t} x(r)drds\right)$$
$$= x(t) - \frac{(1 - \dot{h}(t))}{h(t)}\int_{t-h(t)}^{t} x(s)ds$$
$$- \frac{\dot{h}(t)}{h^2(t)}\int_{t-h(t)}^{t}\int_{s}^{t} x(r)drds,$$

$$\frac{d}{dt}\left(\frac{1}{h - h(t)}\int_{t-h}^{t-h(t)}\int_{s}^{t-h(t)} x(r)drds\right)$$
$$= (1 - \dot{h}(t))x(t - h(t)) - \frac{1}{h - h(t)}\int_{t-h}^{t-h(t)} x(s)ds$$
$$+ \frac{\dot{h}(t)}{(h - h(t))^2}\int_{t-h}^{t-h(t)}\int_{s}^{t-h(t)} x(r)drds.$$

Also, the single and double integral vectors of $\mathbb{M}_1(t - h(t), t)$ and $\mathbb{M}_2(t - h, t - h(t))$ have the zero-equalities (30)-(33) with those of $\mathbb{L}_1(t - h(t), t)$ and $\mathbb{L}_2(t - h, t - h(t))$. Using these relations gives the potential to obtain a less conservative criterion for the time-delay system (1).

Table 2. The maximum admissible upper bound h for Example 2

Method	d			
	0.1	0.2	0.5	0.8
Seuret and Gouaisbaut (2013)	6.590	3.672	1.411	1.275
Kwon et al. (2014)	7.125	4.413	2.243	1.662
Zhi et al. (2017)	7.144	4.462	2.413	1.844
Zhang et al. (2017)	7.230	4.556	2.509	1.940
Theorem 6	7.559	4.925	2.757	2.116

4. NUMERICAL EXAMPLE

This section presents simulation results to verify the effectiveness of the proposed stability criterion.

Example 1. Consider the system (1) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 0.9 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \quad (36)$$

and the time-varying delay $h(t)$ satisfying (2).
For comparison with the existing criteria, this paper calculates the maximum admissible upper bound h for different $d \in \{0.1, 0.2, 0.5, 0.8\}$ and the results are listed in Table 1. From the Table 1, Theorem 6 gives larger h than the existing works.

Example 2. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, A_d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \quad (37)$$

and the time-varying delay $h(t)$ satisfying (2).
For comparison with the existing criteria, this paper calculates the maximum admissible upper bound h for different $d \in \{0.1, 0.2, 0.5, 0.8\}$ and the results are listed in Table 2. From the Table 2, Theorem 6 gives larger h than the existing works.

5. CONCLUSION

This paper proposes a less conservative criterion for systems with time-varying delay by proposing an integral inequality related to the state vector for systems with time-varying delay and exploiting component vectors of the proposed inequality for constructing a Lyapunov-Krasovskii functional. The proposed inequality is based on orthogonal-polynomial-based integral inequality. The component vectors of the proposed inequality have the relation in terms of time-varying delay with those of the orthogonal-polynomial-based integral inequality. Also, the time-derivative of the component vectors of the proposed inequality are represented by those of the orthogonal-polynomial-based integral inequality. The Lyapunov-Krasovskii functional is constructed by utilizing the component vectors of the proposed inequality and the orthogonal-polynomial-based integral inequality. Based on the the Lyapunov-Krasovskii functional, a stability criterion is derived in terms of linear matrix inequalities. Simulation results show that the proposed criterion is less conservative than the criteria in the literature.

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