

# Influence of Time-Delay Mismatch for Robustness and Stability

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**Abstract:** In a practical control system the process always has a time delay. The uncertainty in the knowledge of the time delay i.e. the delay mismatch strongly influences the quality of the closed-loop control. It can cause unwanted instability and influences the robustness of the control. Stability region applicable for this uncertainty is investigated in this paper in connection with the required performance and robustness.

**Keywords:** robustness, stability, performance, time-delay uncertainty

## 1. INTRODUCTION

Identification, control even adaptive algorithms usually assume the apriori knowledge of the process time-delay. This knowledge is sometimes very uncertain and the mismatch coming from a lack of precision in mathematical modeling of the plant and/or changes in the plant parameters with time can result instability. It would be desirable to know how the time-delay mismatch influences the basic robustness and performance behaviors of the closed-loop control.

Some controller design methodologies, mostly for discrete-time systems, include the time-delay of the plant also into the parameters (Bányász and Keviczky, 1994). Unfortunately relatively few papers (e.g., Hocken *et al.*, 1983; Tzypkin, and Fu, 1993) can be found dealing with the influence of the accuracy of the apriori knowledge or estimate of the time-delay, which is sometimes called the time-delay mismatch problem. Our paper investigates the influence of the time-delay uncertainty on the robust stability and performance.

The framework how this issue will be discussed is the *generic two-degree of freedom (GTDOF)* system topology (Keviczky, 2015), which is based on the *Youla-parameterization* (Maciejowski, 1989) providing all realizable stabilizing regulators (*ARS*) for open-loop stable plants and capable to handle the plant time-delay. The advantage of this approach is that it is easy to calculate the "best" reachable optimal regulator depending on the applied  $\mathcal{H}_2$  and/or  $\mathcal{H}_\infty$  norms as criteria. The drawback is that this methodology can be applied only for open-loop stable plants.

A *GTDOF* control system is shown in Fig. 1, where  $y_r, u, y$  and  $w$  are the reference, process input, output and disturbance signals, respectively. The optimal *ARS* regulator of the *GTDOF* scheme (Keviczky and Bányász (1997)) is given by

$$R_o = \frac{P_w K_w}{1 - P_w K_w S} = \frac{Q_o}{1 - Q_o S} = \frac{P_w G_w S_+^{-1}}{1 - P_w G_w S_-^{-1} z^{-d}} \quad (1)$$

where

$$Q_o = Q_w = P_w K_w = P_w G_w S_+^{-1} \quad (2)$$

is the associated optimal *Y-parameter* (Keviczky and Bányász (2002)) furthermore

$$Q_r = P_r K_r = P_r G_r S_+^{-1}; K_w = G_w S_+^{-1}; K_r = G_r S_+^{-1} \quad (3)$$

assuming that the process is factorable as

$$S = S_+ \bar{S}_- = S_+ S_- z^{-d} \quad (4)$$

where  $S_+$  means the inverse stable (*IS*) and  $S_-$  the inverse unstable (*IU*) factors, respectively.  $z^{-d}$  corresponds to the discrete time-delay, where  $d$  is the integer multiple of the sampling time. Here  $P_r$  and  $P_w$  are assumed stable and proper transfer functions (reference models). An interesting result was (Keviczky, and Bányász, 1999) that the optimization of the *GTDOF* scheme can be performed in  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  norm spaces by the proper selection of the serial  $G_r$  and  $G_w$  embedded filters.

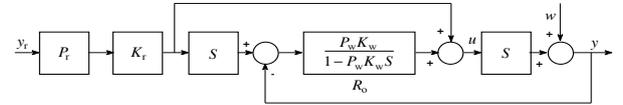


Fig. 1. The *generic TDOF (GTDOF)* control system

## 2. ROBUST STABILITY CONDITIONS FOR *GTDOF* CONTROL SYSTEMS

Be  $M$  the model of the process. Assume that the process and its model are factorizable as

$$S = S_+ \bar{S}_- = S_+ S_- z^{-d}; M = M_+ \bar{M}_- = M_+ M_- z^{-d_m} \quad (5)$$

where  $S_+$  and  $M_+$  mean the inverse stable (*IS*),  $\bar{S}_-$  and  $\bar{M}_-$  the inverse unstable (*IU*) factors, respectively.  $z^{-d}$  and  $z^{-d_m}$  correspond to discrete time delays, where  $d$  and  $d_m$  are the

integer multiple of the sampling time, usually  $d = d_m$  is assumed. (To get a unique factorization it is reasonable to ensure that  $\bar{S}_-$  and  $\bar{M}_-$  are monic, i.e.,  $\bar{S}_-(1) = \bar{M}_-(1) = 1$ , having unity gain.) It is important that the inverse of the term  $z^{-d}$  is not realizable, because it would mean an ideal predictor  $z^d$ . These assumptions mean that  $\bar{S}_- = S_- z^{-d}$  and  $\bar{M}_- = M_- z^{-d_m}$  are uncancelable invariant factors for any design procedure. Introduce the additive

$$\Delta = S - M \quad ; \quad \Delta_+ = S_+ - M_+ \quad ; \quad \Delta_- = S_- - M_- \quad (6)$$

and the relative model errors

$$\ell = \frac{\Delta}{M} = \frac{S - M}{M} \quad ; \quad \ell_+ = \frac{\Delta_+}{M_+} \quad ; \quad \ell_- = \frac{\Delta_-}{M_-} \quad (7)$$

It is easy to show that the characteristic equation using the ARS regulator is (for  $d = d_m = 0$ )

$$M_+ M_- = 0 \quad (8)$$

if a  $Q = \tilde{Q}(M_+ M_-)^{-1}$  parameter is applied, i.e., if someone tries to cancel both factors. This means that the zeros of the  $IU$  factor will appear in the characteristic equation and cause instability. This is why these zeros (and the time delay itself) are called invariant uncancelable factors.

Introducing the model based, nominal complementary sensitivity function

$$\hat{Z} = \frac{\hat{R}M}{1 + \hat{R}M} = \hat{Q}M \quad (9)$$

the well known robust stability condition  $\|\hat{Z}\ell\|_\infty < 1$  for the ARS regulator gives  $\|\hat{Q}M\ell\|_\infty < 1$ , i.e.,

$$|\hat{Q}M| < \frac{1}{|\ell|} \quad \text{or} \quad |\ell| < \frac{1}{|\hat{Q}M|} \quad \forall \omega \quad (10)$$

Thus the robust stability strongly depends on the model  $M$  and how the model-based  $Y$ -parameter  $\hat{Q}$  is selected.

Consider the practical form of the optimal regulator (using  $M$  in (1)) of the GTDOF system based on the available model  $M$  of the process

$$\begin{aligned} \hat{R} &= \frac{P_w G_w M_+^{-1}}{1 - P_w G_w M_- z^{-d_m}} = \\ &= \frac{(P_w G_w M_+^{-1})}{1 - (P_w G_w M_+^{-1})(M_+ M_- z^{-d_m})} = \frac{\hat{Q}}{1 - \hat{Q}M} \end{aligned} \quad (11)$$

where

$$\hat{Q} = P_w G_w M_+^{-1} \quad \text{and} \quad R_o = R(M = S) \quad (12)$$

is the nominal  $Y$ -parameter depending on the model of the plant, which gives back (2) as  $\hat{Q}|_{M=S} = Q_o = P_w G_w S_+^{-1}$ . The dependence on the inverse stable part is direct and visible, however,  $G_w$  generally depends on the inverse unstable part.

We can now state that  $\hat{R}$  is also an ARS controller (but do not forget that only for the model  $M$  and not for the true process  $S$ ).

Analyze the basic robust stability condition (10) obtained for ARS regulators in case of the generic scheme, where the optimal regulator is given by (10) and  $\hat{Q} = P_w G_w M_+^{-1}$  from (11). We get

$$|\hat{Q}M\ell| = |P_w G_w M_+^{-1} M \ell| = |P_w G_w M_- z^{-d} \ell| = |P_w \ell| \quad (13)$$

where  $|G_w M_-| = 1$  can be ensured for a monic  $M_-$  by the optimization of  $G_w$ , furthermore  $|z^{-d}| = 1$ , thus

$$\sup_\omega |\ell| \leq 1/|P_w| \quad \text{or} \quad \|\ell\|_\infty \leq 1/\|P_w\|_\infty \quad (14)$$

Because the right hand side of this inequality depends only on  $P_w$ , which is the reference model for the regulatory property of the GTDOF system, this means that this is a special controller structure, where the performance of the closed-loop is directly influenced by the robustness limit (via the selected  $P_w$ ).

### 3. COMPUTATION OF THE RELATIVE MODEL ERROR

Let us compute the relative model error  $\ell$  for an IS plant, where the model uncertainty comes only from a time-delay mismatch. The delay-free term is assumed to be known exactly, so  $\bar{M}_- = 1$  and  $M_+ = S_+$ . In this case

$$\ell = \ell_d = \frac{\Delta}{M} = \frac{S - M}{M} = \frac{S_+ z^{-d} - S_+ z^{-d_m}}{S_+ z^{-d_m}} = z^{-(d-d_m)} - 1 \quad (15)$$

Assume an equivalent continuous time plant with time-delay  $\tau$  and a model with time-delay  $\tau_m$ . The analogous equivalence means

$$\ell = \ell_\tau = e^{-\Delta\tau s} - 1 \quad (16)$$

where  $\Delta\tau = \tau - \tau_m$ . The robust stability condition (14) for the continuous time case is now

$$\sup_\omega |\ell_\tau| = \sup_\omega |e^{-j\Delta\tau\omega} - 1| \leq 1/|P_w(j\omega)| \quad (17)$$

For the sake of simplicity assume a first order reference model now

$$P_w = \frac{1}{1 + sT_w} \quad ; \quad P_w(j\omega) = \frac{1}{1 + j\omega T_w} \quad (18)$$

which means an  $1/T_w$  bandwidth design goal for the

resulting closed-loop. Using the first order reference model (18) the inequality to be solved for  $\Delta\tau$  is

$$\sup_{\omega} \left| e^{-j\Delta\tau\omega} - 1 \right| \leq |1 + j\omega T_w| \quad (19)$$

which has the solution as a robust stability (RS) condition

$$\ell_{\tau} = \left| \frac{\Delta\tau}{\tau} \right| = \left| 1 - \frac{\tau_m}{\tau} \right| < \frac{\pi}{\sqrt{3}} \frac{T_w}{\tau} = 1.82 \frac{T_w}{\tau} \quad (20)$$

This inequality is one of our major result. The solution of the inequality (19) can be easily followed on Fig. 2.

It is interesting to mention that using the first order Taylor expansion of the exponential term one can get a good approximation of (19) and a sufficient but not necessary condition for small deviations

$$\ell_{\tau} = \left| \frac{\Delta\tau}{\tau} \right| = \left| 1 - \frac{\tau_m}{\tau} \right| < \frac{T_w}{\tau} \quad (21)$$

The interpretation of (20) and (21) is very simple: for small  $T_w$ , which means high closed-loop performance, the model time delay  $\tau_m$  must be close to the true delay  $\tau$ . So it is obtained that the admissible time-delay mismatch is limited by the inverse of the performance. It could be furthermore very interesting how this limit influences the robustness of the loop, see the next section.

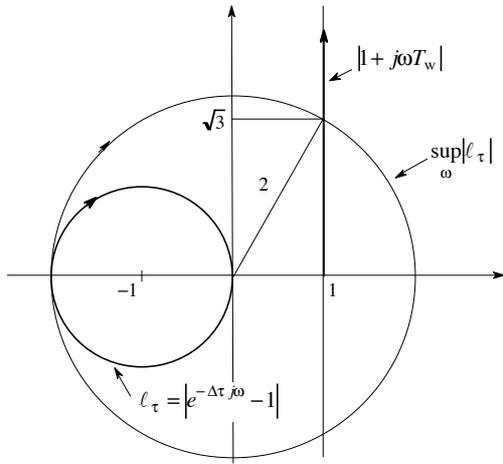


Fig. 2 Simple graphics helping to understand the solution of inequality (19)

There is a simple, however, a somewhat virtual way to increase the robust stability limit (20) by a higher order cutting filter form of the reference model

$$P_w = \frac{1}{(1 + sT_w)^n} \quad ; \quad P_w(j\omega) = \frac{1}{(1 + j\omega T_w)^n} \quad (22)$$

Following the same procedure how (20) was obtained from (19), a more general RS form can be derived

$$\ell_{\tau} = \left| \frac{\Delta\tau}{\tau} \right| = \left| 1 - \frac{\tau_m}{\tau} \right| < a(n) \frac{T_w}{\tau} \quad (23)$$

where the increasing coefficient  $a(n)$  is plotted in Fig. 3.

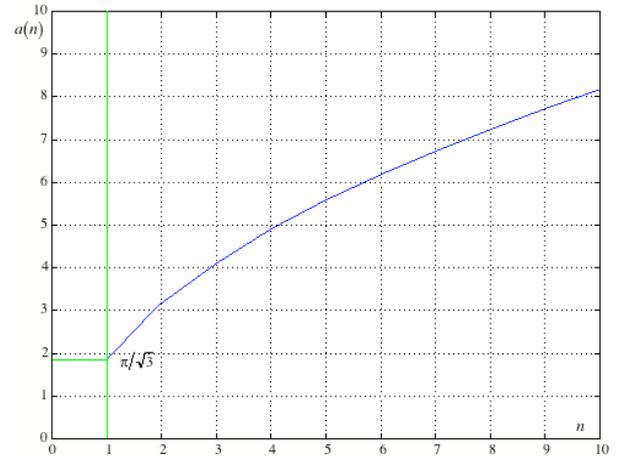


Fig. 3 The  $a(n)$  function

#### 4. PERFORMANCE, ROBUSTNESS AND TIME-DELAY MISMATCH

Detailed investigation of the above mentioned limiting behavior needs further numerical computations. Simple calculations give that the sensitivity function of the GTDOF system with IS plant, having time-delay mismatch for the discrete-time case is (assuming  $G_w = 1$ )

$$E = \frac{1 - P_w z^{-d_m}}{1 + \ell P_w z^{-d_m}} = \frac{1 - P_w z^{-d_m}}{1 + \ell_d P_w z^{-d_m}} \quad (24)$$

and the continuous time equivalent follows as

$$E = \frac{1 - P_w e^{-s\tau_m}}{1 + \ell P_w e^{-s\tau_m}} = \frac{1 - P_w e^{-s\tau_m}}{1 + \ell_t P_w e^{-s\tau_m}} \quad (25)$$

For  $P_w$  given by (18) the sensitivity function (25) becomes

$$E = \frac{1 + sT_w - e^{-s\tau_m}}{1 + sT_w + P_w (e^{-s\tau} - e^{-s\tau_m})} \quad (26)$$

The well-known Nyquist stability margin (the simplest robustness measure) is defined by

$$\begin{aligned} \rho_m &= \rho_{\min}(R) = \min_{\omega} |\rho(\omega, R)| = \min_{\omega} |1 + RS| = \\ &= \min_{\omega} |1 + Y(j\omega)| = \frac{1}{\|E\|_{\infty}} \end{aligned} \quad (27)$$

which is the distance between the point  $(-1+0j)$  and the closest point of the open-loop transfer function  $Y(j\omega)$ . The reciprocal value of the norm is  $\|E\|_{\infty}$ . Unfortunately there is no simple analytical solution to obtain how the closed-loop robustness depends on the time-delay mismatch and on the performance. It is, however, possible to compute the graphical plot of a complex functional relationship  $\rho_m = \rho_{\min}(\tau_m/\tau, T_w/\tau)$  with the help of MATLAB.

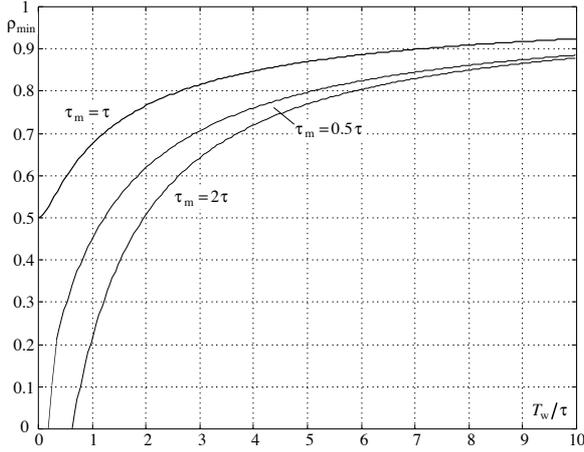


Fig. 4 Function  $\rho_{\min}(T_w/\tau)$  for  $\tau_m = 0.5\tau, \tau, 2\tau$

As a result Fig. 4 shows the function  $\rho_{\min}(T_w/\tau)$  for  $\tau_m = 0.5\tau, \tau, 2\tau$ . For the ideal  $\tau_m = \tau$  (no mismatch) case  $\rho_{\min}$  depends only on our design goal ( $T_w$ ) and on the plant time-delay ( $\tau$ ), more exactly on their relative value  $T_w/\tau$ . The best robustness measure is  $\rho_{\min}(0) = 0.5$  for cases when the reference model  $P_w$  requires a very fast transient response from the time-delay process and the measure is  $\rho_{\min}(\infty) = 1$ , if  $\tau$  is negligible comparing to the time lag of  $P_w$ . It can be well seen that either under- or over-estimation of the time-delay causes considerable decrease of the robustness. Virtually  $\rho_{\min}$  is more sensitive for over-estimation. (The left ends of the plots correspond to the robust stability limit.) While the no mismatch case provides an all stabilizing property for any performance requirement, in case of a non zero time-delay mismatch one can always expect the violation of the robustness stability limit for higher performance design.

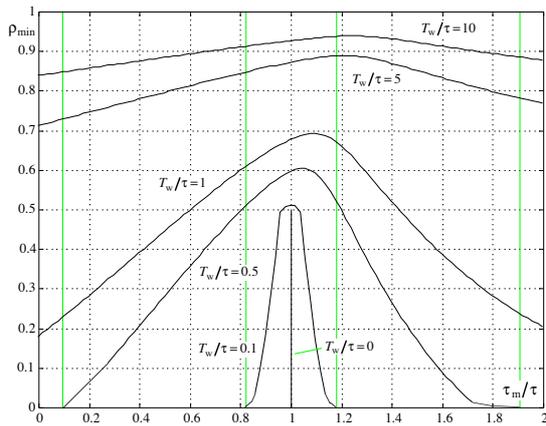


Fig. 5 The function  $\rho_{\min}(\tau_m/\tau)$  parametrized by  $T_w/\tau$

It may be more reasonable to plot the function  $\rho_{\min}(\tau_m/\tau)$  parametrized by  $T_w/\tau$  as Fig. 5 shows (our second major result). One can see how the robustness is extremely sensitive for high performance requirement, when  $T_w/\tau$  is small and how this sensitivity decreases when  $T_w/\tau$  is large for low performance design. It is also interesting to observe, that for small mismatch the over-estimation of the delay gives higher

$\rho_{\min}$ , however, for large mismatch  $\rho_{\min}$  is somewhat more sensitive, as it is shown in Fig. 5.

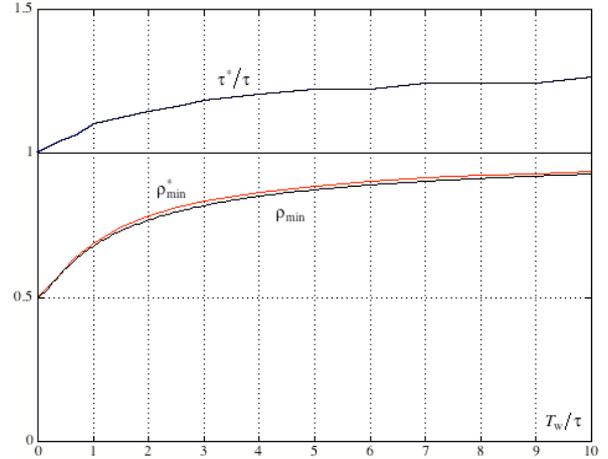


Fig. 6. Influence of the time-delay over-estimation

In a relatively wide range of  $T_w/\tau$ , the over-estimation of the time-delay by  $\tau^*/\tau$  improves (i.e. increases) the  $\rho_{\min}$  to  $\rho_{\min}^*$  according to the maxima of the curves observable in Fig. 5. The over-estimation is less than 25% and the improvement is marginal, less than 5% as Fig. 6 shows.

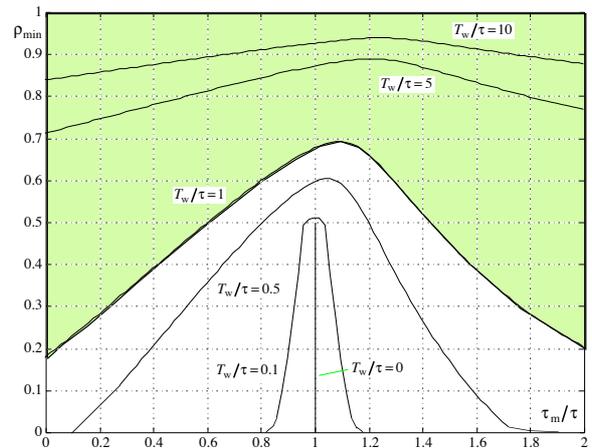


Fig. 7 The shaded area is suggested for acceptable good deal between performance, robustness and time-delay mismatch

If we assume that the time-delay mismatch is less than 20% in a practical case, the robustness degradation is always less than 10% for  $T_w/\tau \geq 0.5$ , which can be well seen in Fig. 6. So if we want to speed up the open-loop process to a time constant, which is considerably less than the delay, then it can only be done using a quite accurate knowledge of the time-delay. Contrary, if someone can expect a considerable variation in the time delay then only a less demanding (slower) design is more reliable and robust.

The above results strengthen the conservative practical design experience that the time-delay is practically equivalent to an  $IU$  zero, i.e. invariant.

It is interesting to summarize the complex relationship between performance, robustness and time-delay uncertainty

and indicate an acceptable area as Fig. 7 shows.

## 5. SIMULATION EXAMPLES

*Example 1.*

The continuous plant is given by the transfer function

$$S(s) = \frac{1}{(1+2s)(1+4s)(1+6s)} e^{-10s}$$

For the YOULA parameterized design separate the transfer function to invertible and non-invertible parts. The non-invertible part of the process is the dead time. The inverse of the invertible part, which is equal to its model:

$$S_+(s) = M_+ = \frac{1}{(1+2s)(1+4s)(1+6s)}$$

Let us choose now the disturbance filter as  $P_w = 1/(1+5s)^3$  and the reference filter as  $P_r = 1/(1+8s)^3$ .  $P_w$  has to be of the same or higher order than  $P_r$ . The YOULA parameter then is

$$Q = P_w M_+^{-1} = \frac{(1+2s)(1+4s)(1+6s)}{(1+5s)^3}$$

In the choice of  $P_w$  the condition of robustness,  $T_w/\tau \geq 0.5$  discussed above was taken into consideration.

It is expected, that from relationship

$$\left| 1 - \frac{\tau_m}{10} \right| < a(3) \frac{1}{10} \approx 0.4$$

acceptable behavior will be reached within mismatch  $6 < \tau_m < 14$

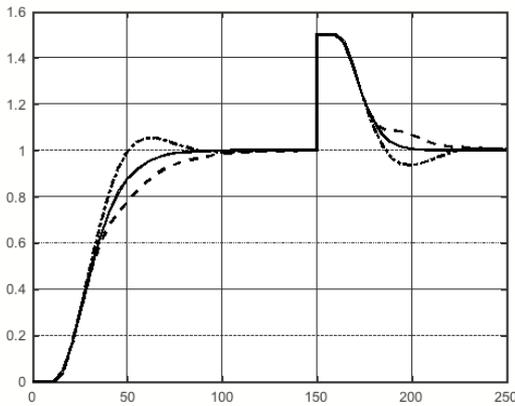


Fig. 8 Step response of the control system with the YOULA controller. Full line: accurate model; dash-dotted line:  $\tau_m = 6$ , dotted line:  $\tau_m = 14$ .

Figure 8 shows the step response and the disturbance rejection of the control system when there is no mismatch between the time delay of the system and its model and in the mismatched cases when the time delay of the model is 6 sec and 14 sec, respectively. A step disturbance of amplitude 0.5 acts at time point 150 sec. It is seen that the control system is robust for these uncertainties in the time delay.

Further simulations show that with this disturbance filter the control system tolerates even much bigger uncertainties in the time delay.

*Example 2.*

The continuous plant is given by the transfer function

$$S(s) = \frac{1}{(1+2s)(1+4s)(1+6s)} e^{-10s}$$

The plant is sampled with sampling time  $T_s = 2$  sec and a zero order hold is applied at its input. Let us design a YOULA parameterized controller. Analyze the effect of different filters.

The pulse transfer function of the plant is

$$G(z) = \frac{0.017792(z+2.396)(z+0.167)}{(z-0.7165)(z-0.6065)(z-0.3679)} z^{-5}$$

Let us separate the pulse transfer function into invertible and non-invertible parts. The dead time cannot be inverted. The zero outside the unit circle cannot be inverted either as it would cause unstable behavior between the sampling points. The second zero is supposed to be in the "good" region" considering Fig. 7. It usually can be cancelled, or if not, it is possible to derive another version of the control algorithm. In the terms of the shift operator  $z^{-1}$  the separation of the pulse transfer function becomes as follows:

$$G_-(z^{-1}) = \frac{(1+2.396z^{-1})z^{-1}}{3.396} z^{-5}$$

(Its static gain has to be 1.)

$$G_+(z^{-1}) = \frac{0.017792 \cdot 3.396 (1+0.167z^{-1})}{(1-0.7165z^{-1})(1-0.6065z^{-1})(1-0.3679z^{-1})}$$

Let us apply now the sampled continuous filters used in the previous example:  $P_w = 1/(1+5s)^3$  and  $P_r = 1/(1+8s)^3$ . Their pulse transfer function is

$$P_r(z) = \frac{0.021615(z+3.098)(z+0.2218)}{(z-0.7788)^3}$$

and

$$P_w(z) = \frac{0.007926(z+2.774)(z+0.1978)}{(z-0.6703)^3}$$

The YOULA parameter with the filters is

$$Q = P_w G_+^{-1} = \frac{0.0079263(z + 2.774)(z + 0.1978)}{(z - 0.6703)^3} \times \\ \times \frac{(z - 0.7165)(z - 0.6065)(z - 0.3679)}{0.017792 \cdot 3.396(z + 0.167)z^2}$$

See some results in Fig. 9.

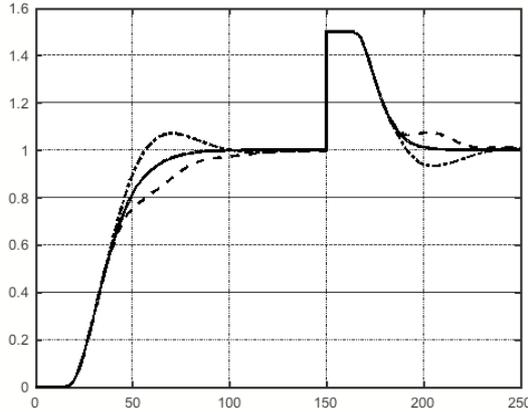


Fig. 9 Step response of the discrete control system with the YOULA controller. Full line: accurate model; dash-dotted line:  $\tau_m = 6$ , dotted line:  $\tau_m = 14$ .

## 6. CONCLUSIONS

Most of the widely applied identification and adaptive control methods assume an a priori known time-delay. It is not easy (although possible) to incorporate the iterative or adaptive estimation of the delay into the recursive methods. Therefore one can always assume a time-delay uncertainty or mismatch at all practical applications. It was discussed here how this mismatch influences the robustness degradation and the reachable closed-loop performance.

A new necessary and sufficient inequality for  $RS$  is derived for the maximum allowable time-delay mismatch and a simpler sufficient condition is also given for a first and an  $n$ -th order reference model.

The complex relationship of robustness, performance and time-delay uncertainty is represented by a special new graphical plot helping the understanding and selection of an acceptable deal between these contradictory criteria.

The investigations show that bandwidth higher than the bandwidth of the delay term ( $T_w < \tau$ ) can be reached only for a considerable lower robustness and at the same time a much more accurate knowledge of the time-delay is necessary. This corresponds to the practical design experience that the corner frequency of a delay term corresponds to an unstable zero, i.e., similarly invariant. So the acceptable performance domain means  $T_w \geq \tau$ .

We found that a certain slight overestimation of the time-

delay improves the robustness, however, a higher overestimation causes considerable robustness degradation again. This observation can be used for model predictive algorithms, too.

Simulation examples demonstrate the effectiveness of the robust design method.

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