Robust stabilisation of linear time-delay systems with uncertainties in the system matrices and in the delay terms

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Abstract: The pseudospectral abscissa is a powerful tool in the evaluation of the robustness of a dynamical system's stability, as it provides a worst-case analysis of the stability of a system when this includes potential uncertainties on parameters. In this paper we consider systems of DDAEs (Delay Differential Algebraic Equations) that incorporate real-valued structured uncertainties on the system matrices and on the delay terms. We propose a robust stabilisation approach for a system of DDAEs, which is based on the optimization of the pseudospectral abscissa with respect to a fixed set of tunable design or controller parameters. Given the non-smoothness nature of the pseudospectral abscissa, standard optimization methods are not efficient and we make use of bundle gradient methods. This approach proves to be very effective, and particularly useful in the design of robust static or dynamic fixed-order controllers for a system of DDEs (Delay Differential Equations) of retarded type. The approach fully exploits the structure on the uncertainty and the property that in applications perturbations are usually real valued, i.e., the uncertainty is not over-bounded and the employed robust stability criterion is necessary and sufficient.

Keywords: Time delay, stability robustness, perturbation analysis, uncertain linear systems, pseudospectral abscissa.

1. INTRODUCTION

The stabilisation of linear time-delay systems is a topic of major concern in control systems theory and a substantial amount of results have contributed to develop and partially solve this problem; without being exhaustive, possible approaches for stabilisation are the use of Lyapunov-Krasovskii functionals and LMI conditions (see for instance Fridman and Shaked (2002); Pepe et al. (2008); Seuret and Johansson (2009)), the direct eigenvalue optimization (Vanbiervliet et al. (2008)), the continuous pole placement method (Michiels et al. (2002)) and the Smith predictor (Palmor (1996); Michiels and Niculescu (2003)). We refer the reader interested in a general overview on the many different stabilisation methods to the monographs by Niculescu (2001), Gu et al. (2003) and Michiels and Niculescu (2007). Controllers used in the stabilisation can also be divided in finite-order and infinite-order controllers. In this work we consider stabilization problems where the closed loop systems takes the form

$$E\dot{x}(t) = A_0(p)x(t) + \sum_{i=1}^{m} A_i(p)x(t - \tau_i(p)), \tag{1}$$

where $E \in \mathbb{R}^{n\times n}$ is allowed to be a singular matrix, $x(t) \in \mathbb{R}^n$ is the state variable, $A_0(p), \ldots, A_m(p) \in \mathbb{R}^{n\times n}$, $0 < \tau_1(p) < \cdots < \tau_m(p)$ are the delay terms, and $p \in \mathbb{R}^{n_p}$ represent the set of design or controller parameters.
Ideally, a dynamical system is stabilized as soon as all the roots of its characteristic equation lie in the open left-half of the complex plane; equivalently, it is stable when its spectral abscissa, i.e. the supremum of the real parts of its characteristic roots, is negative. Therefore, the stabilization of a dynamical system can be carried out by optimizing its spectral abscissa w.r.t some control or design parameter, as explained in Vanbiervliet et al. (2008). However, this method does not guarantee the robustness of the achieved stability condition, and potential uncertainties affecting the system may push one or more eigenvalues to cross the imaginary axis and thus generate an instability in practical applications. Let us then introduce some uncertainties on the matrices coefficients and the delay terms of system (1), which then reads

$$\begin{aligned} E\dot{x}(t) = &\Big(A_0(p) + B_0\delta A_0 C_0\Big)x(t) + \\ &\sum_{i=1}^{m}\Big(A_i(p) + B_i\delta A_i C_i\Big)x\big(t - (\tau_i(p) + \delta\tau_i)\big), \end{aligned} \tag{2}$$

where $B_i,\ \delta A_i,\ C_i$ are real-valued shape matrices of appropriate dimensions for all $i = 0, \ldots, m$, and $\delta\tau_i \in \mathbb{R}$ are such that $|\delta\tau_i| < |\tau_i|$ for all $i = 1, \ldots, m$.
The general forms (1) and (2) may arise as the feedback interconnection of a plant model and a controller with a fixed-order or structure, where p represents the parameter-

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ization of this controller (see below and Section 6 for an example). Note that these general forms also allow to address problems where delays are used as controller parameters. In the following example we show a classic example of an uncertain system with static feedback controller that can be recast using DDAEs.

Example 1. Consider the system with uncertainties

$$\begin{cases} \dot{x}(t) = (A + \delta A)x(t) + (B + \delta B)u(t - \tau + \delta\tau) \\ y(t) = (C + \delta C)x(t), \end{cases} \tag{3}$$

with static controller

$$u(t) = Ky(t). \tag{4}$$

Defining the new state variable $\boldsymbol{\xi} := (x, y, u)^T$, the system can be recast as follows

$$\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{:=\boldsymbol{E}} \dot{\boldsymbol{\xi}}(t) = \left(\underbrace{\begin{bmatrix} A & 0 & 0 \\ C & -I & 0 \\ 0 & K & -I \end{bmatrix}}_{:=\boldsymbol{A_0}} + \underbrace{\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \delta A \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}^T}_{:=\boldsymbol{B_0 \delta A_0 C_0}} \right) \boldsymbol{\xi}(t) +$$

$$+ \left(\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{:=\boldsymbol{A_1}} + \underbrace{\begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \delta C \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}^T}_{:=\boldsymbol{B_1 \delta A_1 C_1}} \right) \boldsymbol{\xi}(t) +$$

$$+ \left(\underbrace{\begin{bmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{:=\boldsymbol{A_2}} + \underbrace{\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \delta B \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}^T}_{:=\boldsymbol{B_2 \delta A_2 C_2}} \right) \boldsymbol{\xi}(t - \tau + \delta\tau).$$

We needed to introduce the zero matrix A_1 because the uncertainties $\delta A,\ \delta C$ are independent one from each other, and they define differently structured perturbations on the A_0 matrix. In the last equation we reformulated the original problem in form of DDAEs with perturbations as in (2), where p is the vectorization of the controller parameters in K. Analogously, we can recast in the general form (2) a system with dynamic controller. Notice that if one substitutes control law (4) in the system of DDAE (3), then the representation (2) is not valid anymore, since the uncertainty δB cannot be decoupled from the controller parameters in K; the adopted reformulation in a DDAE framework then allows us to solve this problem.

Assuming a bound ε on the size of the (constant) uncertainties $\delta A_i,\ \delta\tau_i$ affecting the system, the pseudospectral abscissa is defined as the rightmost eigenvalue generated by all the considered ε-bounded perturbations. Therefore, the pseudospectral abscissa is by definition an adequate measure to evaluate the robustness of a system stability.

The main contribution of this paper is the proposal of a method to robustly stabilise a system of DDAEs by minimizing the pseudospectral abscissa of the system with respect to the fixed set of parameters p. Among the many algorithms in literature for the pseudospectral abscissa computation, we refer the reader to Guglielmi and Lubich (2013) and Michiels and Guglielmi (2012); the former exploits unstructured real-valued perturbation of a standard eigenvalue problem, whereas the latter applies to nonlinear eigenvalue problems (e.g. polynomial and delay eigenvalue problems) whose matrices are perturbed by unstructured complex-valued matrices. The real-valued feature of the former and the nonlinearity structure of the latter are simultaneously taken into account in the method developed in Borgioli et al. (2017), which we adopt in this work. Moreover, this method also takes into account structured perturbations on the system matrices, namely it allows to include an uncertainty on a single coefficient or block of a matrix. It is worth to remark that this algorithm is particularly convenient as real-valued perturbations are more realistic than complex-valued ones, and it also preserves the nonlinear structure of the eigenvalue problem associated with the system. As a second major contribution, in this paper we also include uncertainties affecting the delay terms.

As well known, the pseudospectral abscissa is by definition a continuous but only *almost everywhere* differentiable function, whose local minima are often points of non-differentiability: for this reason, standard optimization points cannot be applied efficiently, and we make use of the HANSO method (Hybrid Algorithm for Nonsmooth Optimization) introduced in Lewis and Overton (2009), that consists in the BFGS method for nonsmooth problems coupled with the gradient sampling method.

The paper is structured as follows: in the following section we formally introduce the uncertainties of the delay eigenvalue problem associated with system (2) and the pseudospectral approach which we use to analyze the robustness of a system stability; in Section 3 we illustrate the method to compute the pseudospectral abscissa; in Section 4 we briefly describe the smoothness properties of the pseudospectral abscissa and provide the derivative of the pseudospectral abscissa w.r.t. design or controller parameters that we use in the optimization; finally in Section 5 we show the applicability of this method to linear delay systems with static or dynamic controllers and present some numerical experiments.

2. THE PSEUDOSPECTRAL APPROACH

In this and the subsequent section we consider the characterization and computation of the pseudospectral abscissa for a fixed value of controller or design parameters p. Therefore, for sake of clarity, we omit the dependence on p in the notations.

We consider here the delay eigenvalue problem associated with the system of perturbed equations (2), where we allow *real-valued, structured* perturbations on the system matrices and *real-valued* perturbations on the delay terms:

$$M(\lambda)y = \Bigg(\lambda E - \big(A_0 + B_0 \delta A_0 C_0\big) - \\ - \sum_{i=1}^{m} \big(A_i + B_i \delta A_i C_i\big) e^{-\lambda\left(\tau_i + \delta\tau_i\right)} \Bigg) y = 0, \tag{5}$$

where $E,\ A_0, \ldots, A_m,\ p,\ \tau_1, \ldots, \tau_m$ are defined as before, $y \in \mathbb{C}^n,\ \lambda \in \mathbb{C},\ B_i,\ \delta A_i,\ C_i$ are real-valued shape matrices of appropriate dimensions for all $i = 0, \ldots, m$, and $\delta\tau_i \in \mathbb{R}$ are such that $|\delta\tau_i| < |\tau_i|$ for all $i = 1, \ldots, m$.

Matrices $B_i,\ C_i$ define the structure of the perturbation on each matrix A_i of the system, that is usually different from one matrix to another. Therefore the size of each matrix δA_i may differ, i.e. $\delta A_i \in \mathbb{R}^{p_i \times q_i},\ i = 0, \ldots, m$. In the following, we will indicate the domain of matrices δA_i as $\mathbb{R}^* := \mathbb{R}^{p_0 \times q_0} \times \cdots \times \mathbb{R}^{p_m \times q_m}$.

We now want to set up a scalar measure of the overall uncertainty affecting our system; we first define the set of all the uncertainties on our system

$$\Delta := \Big(\underbrace{\delta A_0, \ldots, \delta A_m}_{:=\Delta A}, \underbrace{\delta\tau_1, \ldots, \delta\tau_m}_{:=\Delta\tau}\Big),$$

and then, after introducing weights $w_i,\ v_i \in \mathbb{R}^+ \cup \{+\infty\}$, we define the global norm

$$\|\Delta\|_{\text{glob}} := \left\| \begin{bmatrix} w_0 \|\delta A_0\|_F \\ \vdots \\ w_m \|\delta A_m\|_F \\ v_1 |\delta\tau_1| \\ \vdots \\ v_m |\delta\tau_m| \end{bmatrix} \right\|_\infty , \qquad (6)$$

where $\|\cdot\|_F$ indicates the matricial Frobenius norm. From this definition, we naturally obtain that matrices A_i and delay terms τ_i remain unperturbed when the corresponding weight w_i or v_i are set equal to $+\infty$. Moreover an ε-bounded set of perturbations Δ is such that

$$\|\delta A_i\|_F \leq \frac{\varepsilon}{w_i}, \quad |\delta\tau_j| \leq \frac{\varepsilon}{v_j}, \text{ for } \begin{array}{l} i = 0, \ldots, m, \\ j = 1, \ldots, m. \end{array}$$

At this point we can define the *real-valued, structured ε*-pseudospectrum as the following set

$$\Lambda_\varepsilon := \bigcup_{\substack{\Delta \in \mathbb{R}^* \times \mathbb{R}^m \\ \|\Delta\|_{\text{glob}} \leq \varepsilon}} \Big\{ \lambda \in \mathbb{C} : \det M(\lambda) = 0 \Big\}, \qquad (7)$$

and the ε-pseudospectral abscissa function α_ε as

$$\alpha_\varepsilon := \sup\{\Re(\lambda) : \lambda \in \Lambda_\varepsilon\}.$$

In the following, we present the two assumptions on which the rest of the paper grounds: from Michiels (2011) we inherit and adopt throughout the paper the following assumption.

Assumption 2. The matrix $U^T A_0 V$ is nonsingular.

With this assumption, the reformulation of DDAEs as a set of delay differential equations of retarded type coupled with a set of delay difference equations is well-posed. Considering system (1), let $\text{rank}(E) = n - \nu$ and let $U,\ V \in \mathbb{R}^{n \times \nu}$ be respectively a (minimal) basis for the left and right null space of E. Defining

$$\mathbf{U} = [U^\perp\ U],\ \mathbf{V} = [V^\perp\ V] \text{ and } x = \mathbf{V}[x_1^T\ x_2^T]^T,$$

then system (1) can be rewritten in the following form

$$E^{(11)} \dot{x_1}(t) = \sum_{i=0}^m A_i^{(11)} x_1(t - \tau_i) + \sum_{i=0}^m A_i^{(12)} x_2(t - \tau_i),$$
$$0 = \sum_{i=0}^m A_i^{(22)} x_2(t - \tau_i) + \sum_{i=0}^m A_i^{(21)} x_1(t - \tau_i),$$

where we have considered $\tau_0 = 0$ to simplify the notation and where

$$E^{(11)} = {U^\perp}^T E V^\perp$$

and

$$A_i^{(11)} = {U^\perp}^T A_i V^\perp, \quad A_i^{(12)} = {U^\perp}^T A_i V$$
$$A_i^{(21)} = U^T A_i V^\perp, \quad A_i^{(22)} = U^T A_i V, \quad i = 0, \ldots, m.$$

The delay difference equations generate in the spectrum of DDAEs chains of eigenvalues whose imaginary part tend to infinity while the real part has a finite limit; moreover, this limit may be discontinuous with respect to small perturbations in the delay terms. Since in these paper we also deal with uncertainties on the delay terms, it is desirable to have continuity of the spectral abscissa w.r.t. delay terms. This is achieved by making the following assumption, which implies retarded dynamics.

Assumption 3. Matrices $U^T A_i V = 0$ for $i = 1, \ldots, m$.

Essentially, by introducing this assumption we eliminate the aforementioned chains of characteristic roots; as a consequence, in the definition of the pseudospectral abscissa, we can also turn to supremum function into a maximum function. We refer the reader to Michiels (2011) for a detailed description of these dynamics. Of course, these assumptions are also adopted on the perturbed matrices, in order not to switch from retarded to neutral dynamics.

3. COMPUTATION OF THE PSEUDOSPECTRAL ABSCISSA

In this section we illustrate the iterative algorithm for the computation of the pseudospectral abscissa of the perturbed delay eigenvalue problem (5); the purpose of the algorithm is to maximize the real part of the rightmost eigenvalue over all perturbations Δ such that $\|\Delta\|_{\text{glob}} \leq \varepsilon$. The basic idea to reach the maximum is to follow the gradient of the spectral abscissa of the perturbed DEPs in the space of perturbations: thus we consider the gradient flow, which is described by a set of first order ODEs, and we discretize it by using a forward Euler method. However, we present a theoretical result showing that there always exists a smaller class of low-rank perturbations generating a rightmost eigenvalue whose real part is the pseudospectral abscissa. The algorithm then strongly exploits this characterization, by restricting the research to this class of perturbations.

The main theoretical result here presented is obtained as a natural extension of the one presented in Borgioli et al. (2017) to the case including uncertainties on the delay terms. Here and in the following we define the *optimal* matrix perturbations and the *optimal* time-delays perturbations as the perturbations generating the pseudospectral abscissa.

Theorem 4. Let λ_{RM} be the globally rightmost point of the real-valued structured ε-pseudospectrum and assume it is a simple eigenvalue for some ε-bounded perturbation Δ. Then

(i) There always exists a set of perturbations $\widetilde{\Delta A} = (\widetilde{\delta A_0}, \ldots, \widetilde{\delta A_m})$, where $\widetilde{\delta A_i}$ has rank at most two and $w_i \left\| \widetilde{\delta A_i} \right\|_F \leq \varepsilon$ for all $i = 0, \ldots, m$, and for which the rightmost eigenvalue is equal to λ_{RM};

(ii) Let x, y be respectively the unitary left and right eigenvectors of λ_{RM} such that

$$\xi := -x^* \Big(I_n + \big(A_0 + B_0 \delta A_0 C_0\big) + \\ + \sum_{i=1}^m \big(A_i + B_i \delta A_i C_i\big) \tau_i e^{-\lambda(\tau_i + \delta\tau_i)} \Big) y > 0 \qquad (8)$$

and let us define

$$X = [\Re(x)\ \Im(x)],\ Y = [\Re(y)\ \Im(y)],$$

and

$$\Gamma_i = \begin{bmatrix} \Re(e^{-(\tau_i+\delta\tau_i)\lambda_{\text{RM}}}) & -\Im(e^{-(\tau_i+\delta\tau_i)\lambda_{\text{RM}}}) \\ \Im(e^{-(\tau_i+\delta\tau_i)\lambda_{\text{RM}}}) & \Re(e^{-(\tau_i+\delta\tau_i)\lambda_{\text{RM}}}) \end{bmatrix}$$

for $i = 1, \dots, m$, while Γ_0 is the identity matrix with dimension 2. Then for each i, $B_i^T X \Gamma_i Y^T C_i^T$ can be either zero or nonzero. In the latter case, a particular set of optimal perturbations can be expressed as

$$\widetilde{\delta A_i} = -\frac{\varepsilon}{w_i} \frac{B_i^T X \Gamma_i Y^T C_i^T}{\left\| B_i^T X \Gamma_i Y^T C_i^T \right\|_F}, \; i = 0, \dots, m.$$

(iii) Let x, y be defined as before, then

$$\frac{\partial \Re(\lambda_{\mathrm{RM}})}{\partial \delta\tau_i} = \frac{1}{\xi} \Re\Big(x^* \big(A_i + B_i \delta A_i C_i \big) \lambda e^{-\lambda(\tau_i + \delta\tau_i)} y \Big)$$

can also be either zero or nonzero, for $i = 1, \dots, m$. In the latter case, the optimal time-delay perturbations are such that $\widetilde{\delta\tau_i} = \pm \frac{\varepsilon}{v_i}$.

Proof. The proof is an extension to structured perturbations of Theorem 3.3 in Borgioli et al. (2017).

From the characterization here outlined, we can restrict the search for *optimal* matrices and *optimal* delay perturbations respectively to the manifold

$$\begin{aligned} \widehat{\mathcal{S}}_A := \{ (\delta A_0, \dots, \delta A_m) \in \mathbb{R}^* : \operatorname{rank}(\delta A_i) \le 2, \\ w_i \left\| \delta A_i \right\|_F \le \varepsilon, \; i = 0, \dots, m \} \end{aligned} \quad (9)$$

and to the the closed set

$$\widehat{\mathcal{S}}_\tau := \Big[-\frac{\varepsilon}{v_1}, +\frac{\varepsilon}{v_1} \Big] \times \cdots \times \Big[-\frac{\varepsilon}{v_m}, +\frac{\varepsilon}{v_m} \Big]. \quad (10)$$

The algorithm defines a sequence of perturbations of the original eigenvalue problem associated with system (1); these are determined by the couple $(\Delta A, \Delta \tau) \in \widehat{\mathcal{S}} := \widehat{\mathcal{S}}_A \times \widehat{\mathcal{S}}_\tau$ and they are built such that the sequence $\{\Re(\lambda_k)\}_{k \ge 1}$ of the corresponding spectral abscissae is monotonically increasing (where λ_k is the righmost eigenvalue of the perturbed DEP). This is carried out as a discretization of the gradient flow of the spectral abscissa in the space $\widehat{\mathcal{S}}$. Let us consider any continuous path in $\widehat{\mathcal{S}}$, which reads

$$\begin{cases} \delta A_i(t) = -\dfrac{\varepsilon}{w_i} U_i(t) Q_i(t) V_i(t)^T, & t \in \mathbb{R}^+ \\ \delta\tau_i(t) = \dfrac{\varepsilon}{v_i} q_i(t), & t \in \mathbb{R}^+ \end{cases} \quad (11)$$

where the following properties need to be satisfied:

$$\begin{cases} U_i(t)^T \dot{U}_i(t) = 0, & \forall\, t \ge 0 \\ V_i(t)^T \dot{V}_i(t) = 0, & \forall\, t \ge 0 \\ \left\| Q_i(t) \right\|_F \le 1, & \forall\, t \ge 0 \\ |q_i(t)| \le 1, & \forall\, t \ge 0 \end{cases} \quad (12)$$

with $U_i(t) \in \mathbb{R}^{p_i \times 2}, V_i(t) \in \mathbb{R}^{q_i \times 2}$, $Q_i(t) \in \mathbb{R}^{2 \times 2}$ for $i = 0, \dots, m$. An exhaustive justification for the adopted decomposition and the corresponding properties can be found in Borgioli et al. (2017). It is easy to prove that these properties can be imposed via the following differential equations

$$\begin{cases} \dot{U}_i = \Big(I_n - U_i U_i^T \Big) R_i, \\ \dot{V}_i = \Big(I_n - V_i V_i^T \Big) S_i, \\ \dot{Q}_i = \begin{cases} M_i - \langle M_i, Q_i \rangle Q_i, & \text{if } \left\| Q_i \right\|_F = 1, \; \langle M_i, Q_i \rangle > 0 \\ M_i, & \text{otherwise,} \end{cases} \\ \dot{q}_i = \begin{cases} 0 & \text{if } |q_i| = 1, \; r_i q_i > 0 \\ r_i, & \text{otherwise,} \end{cases} \end{cases} \quad (13)$$

where we have introduced the arbitrary quantities $R_i(t) \in \mathbb{R}^{p_i \times 2}$, $S_i(t) \in \mathbb{R}^{q_i \times 2}$, $M_i(t) \in \mathbb{R}^{2 \times 2}$, for $i = 0, \dots, m$ and $r_i(t)$ for $i = 1, \dots, m$, and where we have indicated with $\langle \cdot, \cdot \rangle$ the Frobenius inner product of two matrices. Now, we want to define a specific path where the spectral abscissa λ is proved to be monotonically increasing; to this purpose, we impose the derivative of the spectral abscissa to be nonnegative by making an appropriate choice on the arbitrary quantities R_i, S_i, M_i, r_i. Let us consider the easy case when $\dot{Q}_i(t) = M_i$ and $\dot{q}_i(t) = r_i$. Setting ξ positive as in Equation (8), $\tau_0 = 0$, $v_0 = +\infty$ and omitting parameter t to simplify the notation, the derivative of the real part of the rightmost eigenvalue with respect to t satisfies

$$\begin{aligned} \Re(\dot{\lambda}) = & \sum_{i=0}^{m} \frac{\varepsilon}{\xi w_i} \Re\big(x^* (B_i \dot{U}_i Q_i V_i^T C_i + B_i U_i \dot{Q}_i V_i^T C_i \\ & + B_i U_i Q_i \dot{V}_i^T C_i) e^{-\lambda(\tau_i + \delta\tau_i)} y \big) + \\ & + \sum_{i=0}^{m} \frac{\varepsilon}{\xi v_i} \Re\big(x^* (A_i + B_i \delta A_i C_i) \lambda \dot{q}_i e^{-\lambda(\tau_i + \delta\tau_i)} y \big) = \\ = & \sum_{i=0}^{m} \frac{\varepsilon}{\xi w_i} \big\langle X, B_i (I_n - U_i U_i^T) R_i Q_i V_i^T C_i Y \Gamma_i^T \big\rangle + \\ & + \sum_{i=0}^{m} \frac{\varepsilon}{\xi w_i} \big\langle X, B_i U_i M_i V_i^T C_i Y \Gamma_i^T \big\rangle + \\ & + \sum_{i=0}^{m} \frac{\varepsilon}{\xi w_i} \big\langle X, B_i U_i Q_i S_i^T (I_n - V_i V_i^T) C_i Y \Gamma_i^T \big\rangle + \\ & + \sum_{i=0}^{m} \frac{\varepsilon}{\xi v_i} \underbrace{\Re\big(x^* (A_i + B_i \delta A_i C_i) \lambda e^{-\lambda(\tau_i + \delta\tau_i)} y \big)}_{:= \beta_i} \dot{q}_i = \\ = & \sum_{i=0}^{m} \frac{\varepsilon}{\xi w_i} \big\langle B_i^T X \Gamma_i Y^T C_i^T V_i Q_i^T, (I_n - U_i U_i^T) R_i \big\rangle + \\ & + \sum_{i=0}^{m} \frac{\varepsilon}{\xi w_i} \big\langle U_i^T B_i^T X \Gamma_i Y^T C_i^T V_i, M_i \big\rangle + \\ & + \sum_{i=0}^{m} \frac{\varepsilon}{\xi w_i} \big\langle S_i, (I_n - V_i V_i^T) C_i Y \Gamma_i^T X^T B_i U_i Q_i \big\rangle + \\ & + \sum_{i=0}^{m} \frac{\varepsilon}{\xi v_i} \beta_i \dot{q}_i, \end{aligned}$$

and analogous expressions hold for the other cases in (13). Now, since $(I_n - U_i U_i^T)$, $(I_n - V_i V_i^T)$ are positive semidefinite, $\Re(\dot{\lambda})$ is guaranteed to be nonnegative by the following choices

$$\begin{cases} \dot{U}_i = \Big(I_n - U_i U_i^T \Big) B_i^T X \Gamma_i Y^T C_i^T V_i Q_i^T, \\ \dot{V}_i = \Big(I_n - V_i V_i^T \Big) C_i Y \Gamma_i^T X^T B_i U_i Q_i, \\ \dot{Q}_i = \begin{cases} M_i - \langle M_i, Q_i \rangle Q_i, & \text{if } \left\| Q_i \right\|_F = 1, \; \langle M_i, Q_i \rangle > 0, \\ M_i, & \text{otherwise,} \end{cases} \\ \dot{q}_i = \begin{cases} 0 & \text{if } |q_i| = 1, \; \beta_i q_i > 0 \\ \beta_i, & \text{otherwise,} \end{cases} \end{cases} \quad (14)$$

with $M_i = U_i^T B_i^T X \Gamma_i Y^T C_i^T V_i$. Given these choices for R_i, S_i, M_i, r_i, at each point in $\widehat{\mathcal{S}}$ we are able to follow a direction which ensures an ascent direction of the spectral abscissa λ; therefore, at each iteration in our algorithm we discretize differential equations (14) and perform an Euler forward step to generate a new ε-bounded perturbed eigenvalue problem with a larger spectral abscissa. The boundedness of the pseudospectrum guarantees the convergence of the algorithm, which stops when the derivative $\Re(\dot{\lambda})$ can be approximated to zero. We refer the reader interested in details of the algorithm implementation to Borgioli et al. (2017).

4. SMOOTHNESS PROPERTIES AND OPTIMIZATION OF THE PSEUDOSPECTRAL ABSCISSA

We now take into account the dependence of the nominal matrices A_i and delays τ_i on the parameters p and consider the optimization of the function

$$\begin{aligned} \alpha_\varepsilon(p): \mathbb{R}^{n_p} &\longrightarrow \quad \mathbb{R} \\ p \quad &\longrightarrow \alpha_\varepsilon\ (M(\lambda, p)), \end{aligned}$$

where we have now made explicit the dependence of the perturbed DEP (5) on the controller parameters p.

In order to perform the optimization, let us briefly summarize the pseudospectral abscissa smoothness properties: the spectral abscissa of an eigenvalue problem associated with a system of DDAEs is continuous with respect to matrix coefficients; moreover, under Assumption 3, we can also assume it to be continuous with respect to the delay terms (see Michiels and Niculescu (2007)). Therefore, following from the definition of maximum, the pseudospectral abscissa is also continuous but not everywhere differentiable; typically it is differentiable almost everywhere, and points of non-smoothness are generated by the presence of the maximum function and are characterized by switching of the component of pseudospectrum which contains the globally rightmost eigenvalue. For this reason, the function is often non-differentiable in its local minima.

We perform the optimization using the HANSO algorithm (Overton (2009)), which has been proved to efficiently converge to local minima of nonsmooth, nonconvex functions: example of its applications in this field can be found in Michiels (2011) and Gumussoy and Michiels (2011), where respectively the spectral abscissa and the robust H_∞ norm of a system of DDAEs are optimized.

Next theorem provides the explicit expression of the derivative of α_ε w.r.t. to the design or controller parameters p, whenever this derivative exists.

Theorem 5. Let $\big(\lambda_{\mathrm{RM}}(p), x(p), y(p)\big)$ be respectively the globally rightmost point of Λ_ε and its left and right normalized eigenvectors; for each set of parameter $p \in \mathbb{R}^{n_p}$ let us define $\widetilde{\Delta A}(p) = \big(\widetilde{\delta A_1}(p), \ldots, \widetilde{\delta A_m}(p)\big)$ and $\widetilde{\Delta \tau}(p) = \big(\widetilde{\delta \tau_1}(p), \ldots, \widetilde{\delta \tau_m}(p)\big)$ as the *optimal* matrix perturbations and *optimal* delay perturbations functions such that $\lambda_{\mathrm{RM}}(p)$ is simple; then denoting

$$\zeta := -x^* \Big(E + \sum_{i=0}^{m} (A_i + B_i \widetilde{\delta A_i} C_i) e^{\left(-\lambda_{\mathrm{RM}}(\tau_i + \widetilde{\delta\tau_i})\right)} (\tau_i + \widetilde{\delta\tau_i})\Big) y$$

we can express the derivative of the pseudospectral abscissa w.r.t each parameter p_j as follows

$$\begin{aligned} \frac{d\alpha_\varepsilon}{dp_j} = \frac{\partial \alpha_\varepsilon}{\partial p_j} = \frac{\partial \Re(\lambda_{\mathrm{RM}})}{\partial p_j} = \\ + \frac{1}{\zeta} \Re \Bigg[x^* \Bigg(\sum_{i=0}^{m} \frac{\partial A_i}{\partial p_j} e^{\left(-\lambda_{\mathrm{RM}}(\tau_i + \widetilde{\delta\tau_i})\right)} \Bigg) y - \\ - x^* \Bigg(\sum_{i=0}^{m} (A_i + B_i \widetilde{\delta A_i} C_i) e^{\left(-\lambda_{\mathrm{RM}}(\tau_i + \widetilde{\delta\tau_i})\right)} \lambda_{\mathrm{RM}} \frac{\partial \tau_i}{\partial p_j} \Bigg) y \Bigg], \end{aligned}$$

where we omit the dependence on p to simplify the notation and again assumed $\tau_0 = 0$. Observe that the value of the optimal perturbations $\widetilde{\delta A_i}$, $\widetilde{\delta \tau_i}$ does depend on the set of design or controller parameters p: indeed, although the maximum size of the optimal perturbations is prescribed, the optimal perturbations might assume different values in the parameter space. However, due to the optimality conditions, their derivative w.r.t. p does not affect the derivative of the pseudospectral abscissa.

The main scope of our work is to tune parameters in order to have a negative pseudospectral abscissa. For this reason, convergence to a local but not global minumum of the function does not represent an issue; even more, from a practical point of view, any negative value for the pseudospectral abscissa ensures a robust stability of our system, regardless of the convergence of the algorithm to a local minimum.

HANSO algorithm is intended for unconstrained optimization; however, we might need to force our controller to stay in some region of the parameter space, e.g. we constrain the delay terms to have positive values. For this reason, we also include some continuous penalties in our cost function.

5. NUMERICAL EXPERIMENTS

Here we report an example from literature on which we optimize the pseudospectral abscissa using the procedure described in Section 4 and the software for HANSO available at Overton (2009).

Example 6. We consider system (3) introduced in Example 1, where

$$A = \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}, \quad C = I_3,$$

and $\tau = 5$. This system is unstable, therefore we consider a static feedback $u(t) = Ky(t)$. As in Example 1, we include uncertainties δA, δB on the matrix coefficients and $\delta\tau$ on the time delay τ; we also include an uncertainty δC on the measurement of the state variable. Thus we set

$$\varepsilon = 0.05,\ [w_0, w_1, w_2] = \left[\frac{1}{\|A\|_F}, \frac{1}{\|C\|_F}, \frac{1}{\|B\|_F} \right],\ v_1 = \frac{\varepsilon}{0.5}.$$

With these choices, we allow a maximal relative error of 5% on matrices A, B and C, and we set $|\delta\tau| \leq 0.5$. In Table 1 we report the values of the spectral abscissa α and of the pseudospectral abscissa α_ε in five different cases: in the first case the system is uncontrolled ($K = 0$); in the second case we assume the minimizer for the spectral abscissa $K^s = [0.472\ 0.505\ 0.603]$ (see Vanbiervliet et al. (2008)); in the third case the optimal controller $K^s_\varepsilon = [0.944\ 1.171\ 0.543]$ minimizes the pseudospectral abscissa α_ε; in the last two cases, controllers K^d, respectively K^d_ε, are minimizers for the spectral abscissa, respectively pseudospectral abscissa, of system (3) coupled with a dynamic controller as follows

$$\begin{cases} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t), \\ u(t) &= C_c x_c(t) + D_c y(t), \end{cases}$$

where K^d, K^d_ε are the vectorizations of controller parameters included in A_c, B_c, C_c, D_c, and $x_c(t)$, $u(t) \in \mathbb{R}$. It is worth remarking that $\alpha_\varepsilon(K^s_\varepsilon) < 0$ and $\alpha_\varepsilon(K^d_\varepsilon) < 0$: this means that our optimization process provided an

Table 1. The table shows the spectral abscissa α and the pseudospectral abscissa α_ε in the uncontrolled system ($K = 0$), and in the systems with controllers K^s, K^s_ε, K^d and K^d_ε.

	α	α_ε
Uncontrolled	+1.081e-01	+1.266e-01
K^s	-1.492e-01	+5.550e-02
K^s_ε	-5.663e-02	-3.909e-03
K^d	-2.146e-01	+4.553e-02
K^d_ε	-7.867e-02	-1.490e-02

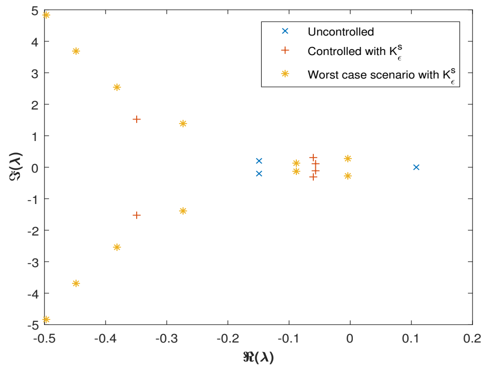

Fig. 1. The figure shows the rightmost eigenvalues for $K = 0$, for $K = K^s_\varepsilon$ and the worst-case scenario for $K = K^s_\varepsilon$.

optimal controller that guarantees the robustness of stability. We also observe that $\alpha_\varepsilon(K^s_\varepsilon) < \alpha_\varepsilon(K^s)$ and that $\alpha(K^s_\varepsilon) > \alpha(K^s)$: this demonstrates the considerably different behaviours of functions α and α_ε and justifies our optimization approach for the robust stabilization: same considerations apply for K^d, K^d_ε. Finally, in Figure 1 we compare the rightmost eigenvalues in the uncontrolled system, in the system controlled with $K = K^s_\varepsilon$ and in the worst-case scenario for $K = K^s_\varepsilon$, where the real part of the globally rightmost eigenvalue is the pseudospectral abscissa.

6. CONCLUDING REMARKS

In this paper we presented an iterative algorithm to compute the pseudospectral abscissa of a system of DDAEs with real-valued structured uncertainties on the system matrices and uncertainties on the delay terms; thanks to the low-rank dynamics exploited, this algorithm also has potential for large-scale systems. We coupled the algorithm with optimization methods for nonsmooth functions in order to minimize the pseudospectral abscissa and thus robustly stabilize the original system. Applications of this method are envisaged in the design of static or dynamic fixed-order controllers for system of DDEs.

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