An Impulsive Goodwin-like Model: The Case of Multiple Delays ⋆

Alexander N. Churilov

Faculty of Mathematics and Mechanics, St. Petersburg State University, Universitetskiy av. 28, Stary Peterhof, 198504, St. Petersburg, Russia, and
Laboratory of Control of Multi-agent, Distributed and Networked Systems, ITMO University, Kronverkskiy av. 49, 197101, St. Petersburg, Russia (e-mail: a_churilov@mail.ru)

Abstract: An impulsive counterpart of the Goodwin biological oscillator with three discrete delays is considered. An impulse-to-impulse discrete map that captures dynamics of the impulsive delayed model is constructed.

Keywords: Impulsive systems, time delays, periodic solutions.

1. INTRODUCTION

A mathematical model called "Goodwin oscillator" was introduced in Goodwin (1965, 1966) and gained significant popularity in mathematical biology (see Murray (2002); Gonze and Abou-Jaoudé (2013); Mackey et al. (2012–2014)). In Smith (1980, 1983) the Goodwin's model was applied to mathematical endocrinology, namely, to describe oscillations of hormones' levels in the male reproductive hormonal axis. The specific of endocrinological regulation systems that involve the hypothalamus is that hypothalamic hormones are released impulsively, governed by ensembles of neurones. The conventional Goodwin model does not suit properly to describe this impulsive effect. To improve adequacy of the model Churilov et al. (2009) proposed to substitute ordinary differential equations presented in the Goodwin's scheme for impulsive (functional-differential) equations used in applied mathematics (see Lakshmikantham et al. (1989); Bainov and Simeonov (1993); Samoilenko and Perestyuk (1995); Stamova and Stamov (2016)). In electrical engineering such systems are known as pulse modulated (see e. g. Jones et al. (1961); Gelig and Churilov (1998)).

The Goodwin model with time delays was studied in a number of works Smith (1983); Cartwright and Husain (1986); Das et al. (1994); Mukhopadhyay and Bhattacharyya (2004); Ren (2004); Enciso and Sontag (2004); Efimov and Fradkov (2007); Greenhalg and Khan (2009); Li (2015). Beginning with Churilov et al. (2012) a time delay was introduced into the impulsive Goodwin model. The main mathematical tool that was employed was a construction of a discrete impulse-to-impulse map (called the Poincaré map in the theory of hybrid systems, see Haddad et al. (2006)). This map can also be considered as a special translation operator along the trajectories of a continuous-time system Krasnoselskii (1968). Poincaré maps preserve most of the properties of the initial system and can be easily implemented in computer programs. They are especially useful for finding periodic solutions and bifurcation analysis. The basic property of a delayed system that allows to construct Poincaré maps effectively was called finite-dimensional reducibility. It was firstly introduced in Churilov et al. (2012) and developed in a number of subsequent publications, see Churilov et al. (2013, 2014a); Zhusubaliyev et al. (2014); Churilov et al. (2014b); Churilov and Medvedev (2014); Zhusubaliyev et al. (2015); Churilov and Medvedev (2016); Churilov et al. (2016).

The Goodwin model with multiple delays was studied in Liu and Deng (1991); Cao and Jiang (2011); Huang and Cao (2015); Zhang et al. (2015); Sun et al. (2016). In endocrine systems multiple delays describe times required to transport hormones from one organ to another through the bloodstream, and also times for hormones' synthesis. In Churilov et al. (2017) an impulsive counterpart of the three-dimensional Goodwin oscillator was considered with the help of an impulse-to-impulse map. In this paper the same approach is extended to a Goodwin-like impulsive system of a higher dimension that necessitates reworking the proof of the main statement (Theorem 4).

2. FD-REDUCIBILITY OF A LINEAR SYSTEM WITH TWO DISCRETE DELAYS

Let us consider a linear system with two discrete delays

$$u' = U_1 u(t), \tag{1}$$

$$v' = U_2 v(t) + G_1 u(t-\tau_1) \tag{2}$$

$$w' = U_3 w(t) + G_2 v(t-\tau_2) \tag{3}$$

for $t \geqslant t_0$. Here U_1, U_2, U_3, G_1, G_2 are constant matrix blocks whose sizes are $p_1 \times p_1$, $p_2 \times p_2$, $p_3 \times p_3$, $p_2 \times p_1$, $p_3 \times p_2$, respectively, and $u(\cdot)$, $v(\cdot)$, $w(\cdot)$ are functions with vector values of dimensions p_1, p_2, p_3.

Let us introduce a p-dimensional vector

$$x(\cdot) = \mathrm{col}\{u(\cdot),\ v(\cdot),\ w(\cdot)\}$$

⋆ The work was supported by the Russian Foundation for Basic Research (Grant 17-01-00102-a) and the Government of the Russian Federation (Grant 08-08).

with $p=p_1+p_2+p_3$. Then system (1)–(3) can be rewritten in a matrix form as

$$x' = A_0x(t) + A_1x(t-\tau_1) + A_2x(t-\tau_2), \tag{4}$$

where A_0, A_1, A_2 are constant $p\times p$ matrices,

$$A_0 = \begin{bmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 0 & 0 \\ G_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & G_2 & 0 \end{bmatrix}. \tag{5}$$

The initial problem for (4) can be defined as

$$\begin{aligned} u(t) &= \varphi_u(t), \quad t_0-\tau_1 \leqslant t \leqslant t_0, \\ v(t) &= \varphi_v(t), \quad t_0-\tau_2 \leqslant t \leqslant t_0, \\ w(t_0) &= w_0, \end{aligned}$$

where $\varphi_u(\cdot)$, $\varphi_v(\cdot)$ are initial functions and w_0 is a constant vector.

Definition 1. A time-delay linear system (4) is called *finite-dimensional reducible (FD-reducible)* if for every t_0 there exist a constant $p\times p$ matrix D and a positive number $\bar\tau$ such that any solution $x(t)$ of (4) defined for $t \geqslant t_0$ satisfies the ordinary differential equation

$$x' = Dx(t) \tag{6}$$

for $t \geqslant t_0+\bar\tau$. The matrix D and the number $\bar\tau$ are independent of the initial data.

Certainly, the property of FD-reducibility is rather restrictive, but it holds for linear systems with matrices A_0, A_1, A_2 having a special "cyclic" structure (5).

Lemma 1. System (4) with coefficients (5) is FD-reducible with $\bar\tau = \tau_1+\tau_2$ and the matrix D can be defined as

$$D = D_0 + A_2 \mathrm{e}^{-D_0\tau_2} \quad \text{with} \quad D_0 = A_0 + A_1\mathrm{e}^{-A_0\tau_1}. \tag{7}$$

Proof. Since (1) is independent of (2), (3), we have

$$u(t) = \mathrm{e}^{U_1(t-t_0)}u(t_0), \quad t\geqslant t_0,$$

and hence

$$u(t-\tau_1) = \mathrm{e}^{-U_1\tau_1}u(t), \quad t \geqslant t_0+\tau_1.$$

Thus (1), (2) can be rewritten as

$$u' = U_1u(t), \quad v' = U_2v(t) + G_1\mathrm{e}^{-U_1\tau_1}u(t), \quad t\geqslant t_0+\tau_1.$$

In a matrix notation we have

$$z' = D_z z(t), \quad t \geqslant t_0+\tau_1,$$

where

$$D_z = \begin{bmatrix} U_1 & 0 \\ G_1\mathrm{e}^{-U_1\tau_1} & U_2 \end{bmatrix}, \quad z(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}. \tag{8}$$

This implies

$$z(t-\tau_2) = \mathrm{e}^{-D_z\tau_2}z(t), \quad t\geqslant t_0+\tau_1+\tau_2.$$

From (3) it follows

$$w' = U_3w(t) + [0 \quad G_2]z(t-\tau_2)$$

Introduce a $p_3\times(p_1+p_2)$ matrix

$$G_z = [0 \quad G_2]\mathrm{e}^{-D_z\tau_2}. \tag{9}$$

Hence

$$\begin{aligned} z' &= D_z z(t), \\ w' &= U_3w(t) + G_z z(t), \quad t\geqslant t_0+\tau_1+\tau_2. \end{aligned}$$

Because $x(\cdot) = \mathrm{col}\{z(\cdot), w(\cdot)\}$, (6) is satisfied with

$$D = \begin{bmatrix} D_z & 0 \\ G_z & U_3 \end{bmatrix}.$$

It is easily seen that

$$D_0 = \begin{bmatrix} D_z & 0 \\ 0 & U_3 \end{bmatrix}, \quad \mathrm{e}^{-D_0\tau_2} = \begin{bmatrix} \mathrm{e}^{-D_z\tau_2} & 0 \\ 0 & \mathrm{e}^{-U_3\tau_2} \end{bmatrix},$$

which implies (7). □

Notice that

$$\det(sI_p - A_0 - A_1\mathrm{e}^{-A_0\tau_1} - A_2\mathrm{e}^{-A_0\tau_2}) = \det(sI-A_0)$$

for all complex s. Thus the spectrum of the linear system (4) is finite and independent of τ_1, τ_2.

3. IMPULSIVE MODEL WITH THREE DELAYS

In this section we consider an impulsive system whose continuous part has the form of (4) including two discrete delays τ_1 and τ_2. The third discrete delay τ_3 is incorporated into the discrete part of the system.

We shall deal with an impulsive system

$$\begin{aligned} x' &= A_0x(t) + A_1x(t-\tau_1) + A_2x(t-\tau_2), \\ &\quad t_n < t < t_{n+1}, \end{aligned} \tag{10}$$

$$\sigma(t) = Cx(t), \tag{11}$$

$$x(t_n^+) - x(t_n^-) = \lambda_n B, \quad n=0,1,\ldots, \tag{12}$$

$$\begin{aligned} t_{n+1} &= t_n + T_n, \quad T_n = \Phi(\sigma(t_n-\tau_3)), \\ \lambda_n &= F(\sigma(t_n-\tau_3)). \end{aligned} \tag{13}$$

Assume that its matrix coefficients have specific block forms (5) and

$$B = \mathrm{col}\{B_u,\ 0,\ 0\}, \quad C = [0 \quad 0 \quad C_w]. \tag{14}$$

Thus B_u is $p_1\times 1$ and C_w is $1\times p_3$.

Here $\Phi(\cdot)$, $F(\cdot)$ are continuous $\mathbb{R}\to\mathbb{R}$ functions with bounds

$$0 < \Phi_1 \leqslant \Phi(\cdot) \leqslant \Phi_2, \quad 0 < F_1 \leqslant F(\cdot) \leqslant F_2, \tag{15}$$

where Φ_i, F_i, $i=1,2$, are positive constants. In particular, inequality $\Phi_1 > 0$ implies that the sequence $\{t_n\}_{n=0}^{\infty}$ has no accumulation points and $t_n\to+\infty$ as $n\to\infty$.

As previously, $x(\cdot) = \mathrm{col}\{u(\cdot),\ v(\cdot),\ w(\cdot)\}$. Thus equation (10) can be rewritten as (1)–(3). Then $\sigma(t) = C_w w(t)$. Evidently, only the function $u(t)$ has jumps

$$u(t_n^+) - u(t_n^-) = \lambda_n B_u, \tag{16}$$

while the functions $v(t)$, $w(t)$, $\sigma(t)$ are continuous. Formulas (11)–(13) describe an amplitude-frequency pulse modulation with a modulating signal $\sigma(t)$ (see Gelig and Churilov (1998)). The right-hand side of system (10) has jumps at the points t_n, $t_n+\tau_1$, $t_n+\tau_2$, $n\geqslant 0$, while in the time intervals between these points system (10) is linear.

System (10)–(13) is subject to initial conditions given by the continuous initial functions $\varphi_u(\cdot)$, $\varphi_v(\cdot)$, $\varphi_w(\cdot)$. More precisely,

$$\begin{aligned} u(t) &= \varphi_u(t), \quad t_0-\tau_1 \leqslant t < t_0, \\ v(t) &= \varphi_v(t), \quad t_0-\tau_2 \leqslant t \leqslant t_0, \\ w(t) &= \varphi_w(t), \quad t_0-\tau_3 \leqslant t \leqslant t_0, \end{aligned}$$

$$T_0 = \Phi(C_w\varphi_w(t_0-\tau_3)), \quad \lambda_0 = F(C_w\varphi_w(t_0-\tau_3))$$

and $u(t_0^+) - u(t_0^-) = \lambda_0 B_u$. Notice that the values of $\varphi_w(t)$ inside the interval $t_0-\tau_3 < t < t_0$ do not influence the solution and can be chosen arbitrarily.

We will use notation (7), (8) for $z(\cdot)$, D_z, G_z, D_0, D introduced in the previous section.

The following two assumptions are made with respect to the time delay values:

$$\tau_2 > \tau_1, \quad \Phi_1 > \tau_1 + \tau_2 + \tau_3. \tag{17}$$

The second inequality in (17) implies that the lower bound for the length of a sampling interval

$$T_n > \tau_1 + \tau_2 + \tau_3 \tag{18}$$

for all solutions and all $n \geqslant 0$.

For D given in (7) introduce a row vector

$$\tilde{C} = C\mathrm{e}^{-D\tau_3}. \tag{19}$$

Introduce the shorthand notation $\bar{x}_n = x(t_n^-)$.

Lemma 2. Under assumption (17) the formulas for T_n, λ_n in (13) can be rewritten for $n \geqslant 1$ as

$$T_n = \Phi(\tilde{C}\bar{x}_n), \quad \lambda_n = F(\tilde{C}\bar{x}_n), \tag{20}$$

where $\tilde{C}$ is defined by (19).

Proof. As proved in Lemma 1, the continuous part of the system given by (10) is FD-reducible for $t \geqslant \tau_1 + \tau_2$. Hence $x(t)$ satisfies

$$x' = Dx(t) \quad \text{for} \quad t_n + \tau_1 + \tau_2 < t < t_{n+1}, \quad n = 0, 1, 2, \ldots. \tag{21}$$

It follows from (18) that $t_n - \tau_3 > t_{n-1} + \tau_1 + \tau_2$ for all $n \geqslant 1$. Then (21) implies $x(t_n - \tau_3) = \mathrm{e}^{-D\tau_3}\bar{x}_n$. Thus

$$Cx(t_n - \tau_3) = C\mathrm{e}^{-D\tau_3}\bar{x}_n = \tilde{C}\bar{x}_n \quad \text{for all} \quad n = 1, 2, \ldots.$$

Then for $n \geqslant 1$ the expressions in (13) can be rewritten as (20). □

Finally, notice that the next statement can be obtained immediately.

Theorem 3. Under supposition (15) the following properties of system (10)–(13) are valid.

(i) System (10)–(13) has no equilibria.

(ii) Let the matrices U_1, U_2, U_3 be Hurwitz stable. Then for any solution of (10)–(13) the Euclidean norm $\|x(t)\|$ is bounded for $t \geqslant t_0$.

(iii) Assume that the matrices U_1, U_2, U_3 are Metzler, i. e. all their nondiagonal elements are nonnegative. Suppose that all the elements of the matrices G_1, G_2, B_u are also nonnegative. Then system (10)–(13) is positive (see Luenberger (1979)). This means that if the initial functions $\varphi_u(\cdot)$, $\varphi_v(\cdot)$, $\varphi_w(\cdot)$ are elementwise nonnegative, then all the components of $x(t)$ are also nonnegative for $t \geqslant t_0$.

4. REDUCTION TO A DISCRETE-TIME SYSTEM

In this section we will obtain explicit formulas for the solutions of (10)–(13).

For brevity introduce notation

$$t_n^* = t_n + \tau_1, \quad \hat{t}_n = t_n + \tau_2, \quad t_n^{**} = t_n + \tau_1 + \tau_2.$$

From (17), (18) it follows that

$$t_n < t_n^* < \hat{t}_n < t_n^{**} < t_{n+1}.$$

Theorem 4. For $t > t_1$ any solution of (10)–(13) satisfies a delay-free equation

$$x' = Dx(t) - (D - A_0)\alpha(t) - (D - D_0)\beta(t). \tag{22}$$

Here on every interval $t_n < t < t_{n+1}$, $n \geqslant 1$, the functions $\alpha(t)$, $\beta(t)$ are defined as

$$\alpha(t) = \begin{cases} \lambda_n \mathrm{e}^{A_0(t-t_n)}B, & t_n < t < t_n^*, \\ 0, & t_n^* < t < t_{n+1}, \end{cases} \tag{23}$$

$$\beta(t) = \begin{cases} 0, & t_n < t < t_n^*, \\ \lambda_n \mathrm{e}^{D_0(t-t_n^*)}\mathrm{e}^{A_0\tau_1}B, & t_n^* < t < t_{**}, \\ 0, & t_n^{**} < t < t_{n+1}. \end{cases} \tag{24}$$

Proof. The proof of Theorem 4 is lengthy and is given in Appendix.

Theorem 4 is the key result of this paper. Equations (22), (23), (24) present a reduction of the initial impulsive system with delays to delay-free equations, however only for $t \geqslant t_1$.

By integrating (22) we can get explicit formulas for $x(t)$.

Corollary 1. Any solution $(t_n, x(t))$ of (10)–(13) obeys for all $t_n < t < t_{n+1}$, $n \geqslant 1$, the following relationships

$$x(t) = \mathrm{e}^{D(t-t_n)}\bar{x}_n + \lambda_n\theta(t)B, \tag{25}$$

where

$$\theta(t) = \begin{cases} \mathrm{e}^{A_0(t-t_n)}, & t_n < t \leqslant t_n^*, \\ \mathrm{e}^{D_0(t-t_n^*)}\mathrm{e}^{A_0\tau_1}, & t_n^* < t \leqslant t_n^{**}, \\ \mathrm{e}^{D(t-t_n^{**})}\mathrm{e}^{D_0\tau_2}\mathrm{e}^{A_0\tau_1}, & t_n^{**} < t < t_{n+1}. \end{cases} \tag{26}$$

Proof. From Theorem 4 it follows that any solution $(t_n, x(t))$ of (10)–(13) obeys the delay-free equation (22). Let us demonstrate that by integrating (22) we will come to (25).

Case (i). Let $t_n < t < t_n^*$. Theorem 4 yields (22) with $\alpha(t) = \lambda_n \mathrm{e}^{A_0(t-t_n)}B$, $\beta(t) \equiv 0$. Obviously, $\alpha' = A_0\alpha(t)$. Introduce a difference $y(t) = x(t) - \alpha(t)$. From (22) it follows

$$\begin{aligned} y' &= Dy(t), \quad t_n < t < t_n^*, \\ y(t_n^+) &= x(t_n^+) - \lambda_n B = \bar{x}_n. \end{aligned} \tag{27}$$

Integrating (27), we obtain

$$y(t) = \mathrm{e}^{D(t-t_n)}\bar{x}_n, \quad t_n < t < t_n^*.$$

Since $x(t) = y(t) + \alpha(t)$, we come to the first case in (25), (26). Notice that by setting $t = t_n^*$ from the first formula of (26) we have

$$x(t_n^*) = \mathrm{e}^{D\tau_1}\bar{x}_n + \lambda_n \mathrm{e}^{A_0\tau_1}B. \tag{28}$$

Case (ii). Let $t_n^* < t < t_n^{**}$. From Theorem 4 we obtain (22) with $\alpha(t) = 0$, $\beta(t) = \lambda_n \mathrm{e}^{D_0(t-t_n^*)}\mathrm{e}^{A_0\tau_1}B$. Hence

$$\beta' = D_0\beta(t), \quad \beta(t_n^*) = \lambda_n \mathrm{e}^{A_0\tau_1}B. \tag{29}$$

From (22), (28) and (29) we conclude that the difference $y(t) = x(t) - \beta(t)$ satisfies

$$\begin{aligned} y' &= Dy(t), \quad t_n^* < t < t_n^{**}, \\ y(t_n^*) &= \mathrm{e}^{D\tau_1}\bar{x}_n. \end{aligned} \tag{30}$$

Integrating (30) and using $x(t) = y(t) + \beta(t)$, we get the second case in (25), (26). Moreover, we have

$$x(t_n^{**}) = \mathrm{e}^{D(\tau_1+\tau_2)}\bar{x}_n + \lambda_n \mathrm{e}^{D_0\tau_2}\mathrm{e}^{A_0\tau_1}B. \tag{31}$$

Case (iii). Let $t_n^{**} < t < t_{n+1}$. Then $x' = Dx(t)$ and

$$x(t) = \mathrm{e}^{D(t-t_n^{**})}x(t_n^{**}), \quad t_n^{**} < t < t_{n+1}. \tag{32}$$

From (31) and (32) we come to the third case in (25), (26). □

It can be concluded from Corollary 1 that the dynamics of (10)–(13) at points t_n, $n = 1, 2, \dots$, obey the following discrete (Poincar'e) map.

Corollary 2. For any solution $(t_n, x(t))$ of (10)–(13) it holds for $n = 1, 2, \dots$ that

$$\bar{x}_{n+1} = Q(\bar{x}_n), \tag{33}$$

where

$$Q(\bar{x}) = \mathrm{e}^{D\Phi(\tilde{C}\bar{x})}\bar{x} + F(\tilde{C}\bar{x})\mathrm{e}^{D(\Phi(\tilde{C}\bar{x})-\tau_1-\tau_2)}\mathrm{e}^{D_0\tau_2}\mathrm{e}^{A_0\tau_1}B.$$

Proof. With $t = t_{n+1}$ formulas (31) and (32) imply

$$\bar{x}_{n+1} = \mathrm{e}^{DT_n}\bar{x}_n + \lambda_n\mathrm{e}^{D(T_n-\tau_1-\tau_2)}\mathrm{e}^{D_0\tau_2}\mathrm{e}^{A_0\tau_1}B.$$

Thus (33) follows. $\square$

Thus given two initial points $\bar{x}_0$, $\bar{x}_1$, the subsequent points $\bar{x}_2$, $\bar{x}_3, \dots$ can be found by recurrence (33). The initial functions $\varphi_u(t)$, $\varphi_v(t)$, $\varphi_w(t)$, are necessary to calculate the values of $\bar{x}_0$, $\bar{x}_1$, but they do not influence further values $\bar{x}_n$.

Obviously, the function $Q(\cdot)$ is continuous as a composition of continuous functions. If the functions $\Phi(\cdot)$, $F(\cdot)$ are smooth, then $Q(\cdot)$ is also smooth.

5. CONCLUSION

An impulsive delayed system whose continuous part has a specific "cyclic" structure is considered. It is shown that solutions of this system calculated at sampling times (beginning with the third sample) obey a discrete-time equation (an impulse-to-impulse map) (33) that is independent on the initial data (Corollary 2). Additionally, explicit formulas for solutions are provided for times taken inside a sampling period (Corollary 1). Thus the impulse-to-impulse map thoroughly characterizes solutions of the continuous-time equation with time delays. The discrete-time equation (33) can be readily explored by means of a computer modelling (examples of such analysis are given in Churilov et al. (2017)). The results of this paper generalize those of Churilov et al. (2017) for a multidimensional case.

ACKNOWLEDGEMENTS

The author acknowledges Alexander Medvedev, Uppsala University, Sweden, and Zhanybai Zhusubaliyev, Southwest State University, Russia, for cooperation and discussion.

REFERENCES

Bainov, D. and Simeonov, P. (1993). *Impulsive Differential Equations: Periodic Solutions and Applications.* Longman, Harlow, UK.

Cao, J. and Jiang, H. (2011). Stability and Hopf bifurcation analysis on Goodwin model with three delays. *Chaos Solitons Fractals*, 44, 613–618.

Cartwright, M. and Husain, M. (1986). A model for the control of testosterone secretion. *J. Theor. Biol.*, 123, 239–250.

Churilov, A.N. and Medvedev, A. (2014). An impulse-to-impulse discrete-time mapping for a time-delay impulsive system. *Automatica*, 50(8), 2187–2190.

Churilov, A.N. and Medvedev, A. (2016). Discrete-time map for an impulsive Goodwin oscillator with a distributed delay. *Math. Control Signals Syst.*, 28(1), 1–22.

Churilov, A.N., Medvedev, A., and Mattsson, P. (2012). Periodical solutions in a time-delay model of endocrine regulation by pulse-modulated feedback. In *Proc. 51st IEEE Conf. Decis. Control*, 362–367. Maui, Hawaii.

Churilov, A.N., Medvedev, A., and Mattsson, P. (2013). Finite-dimensional reducibility of time-delay systems under pulse-modulated feedback. In *Proc. 52nd IEEE Conf. Decis. Control*, 2078–2083. Florence, Italy.

Churilov, A.N., Medvedev, A., and Mattsson, P. (2014a). Periodical solutions in a pulse-modulated model of endocrine regulation with time-delay. *IEEE Trans. Automat. Control*, 59(3), 728–733.

Churilov, A.N., Medvedev, A., and Shepeljavyi, A.I. (2009). Mathematical model of non-basal testosterone regulation in the male by pulse modulated feedback. *Automatica*, 45(1), 78–85.

Churilov, A.N., Medvedev, A., and Zhusubaliyev, Zh.T. (2014b). Periodic modes and bistability in an impulsive Goodwin oscillator with large delay. In *19th World Congress IFAC Proc.*, 3340–3345. Cape Town, South Africa.

Churilov, A.N., Medvedev, A., and Zhusubaliyev, Zh.T. (2016). Impulsive Goodwin oscillator with large delay: periodic oscillations, bistability, and attractors. *Nonlinear Anal. Hybrid Syst.*, 21, 171–183.

Churilov, A., Medvedev, A., and Zhusubaliyev, Z. (2017). Discrete-time mapping for an impulsive Goodwin oscillator with three delays. *Int. J. Bifurcation Chaos*, 27(12), 1750182 [17 pages].

Das, P., Roy, A.B., and Das, A. (1994). Stability and oscillations of a negative feedback delay model for the control of testosterone secretion. *BioSystems*, 32(1), 61–69.

Efimov, D. and Fradkov, A. (2007). Oscillatority conditions for nonlinear systems with delay. *J. Appl. Math.*, 2007. Article ID 72561.

Enciso, G. and Sontag, E.D. (2004). On the stability of a model of testosterone dynamics. *J. Math. Biol.*, 49, 627–634.

Gelig, A.Kh. and Churilov, A.N. (1998). *Stability and Oscillations of Nonlinear Pulse-modulated Systems.* Birkhäuser, Boston.

Gonze, D. and Abou-Jaoudé, W. (2013). The Goodwin model: Behind the Hill function. *PLoS ONE*, 8(8), e69573.

Goodwin, B.C. (1965). Oscillatory behavior in enzymatic control processes. In G. Weber (ed.), *Adv. Enzyme Regul.*, volume 3, 425–438. Pergamon, Oxford.

Goodwin, B.C. (1966). An intrainment model for timed enzyme synthesis in bacteria. *Nature*, 209(5022), 479–481.

Greenhalg, D. and Khan, Q.J.A. (2009). A delay differential equation mathematical model for the control of the hormonal system of the hypothalamus, the pituitary and the testis in man. *Nonlin. Anal.*, 71, e925–e935.

Haddad, W.M., Chellaboina, V., and Nersesov, S.G. (2006). *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control.* Princeton Univ. Press, Princeton and Oxford.

Huang, C. and Cao, J. (2015). Hopf bifurcation in an n-dimensional Goodwin model via multiple delays feedback. *Nonlinear Dynam.*, 79(4), 2541–2552.

Jones, R., Li, C., Meyer, A., and Pinter, R. (1961). Pulse modulation in physiological systems, phenomenological aspects. *IRE Trans. Biomed. Electron.*, 8(1), 59–67.

Krasnoselskii, M.A. (1968). *The Operator of Translation along the Trajectories of Differential Equations.* Amer. Math. Soc., Providence, RI.

Lakshmikantham, V., Bainov, D.D., and Simeonov, P.S. (1989). *Theory of Impulsive Differential Equations.* World Scientific, Singapore.

Li, X. (2015). Global existence of periodic solutions in a physiological model with delay. *Acta Math. Appl. Sinica, English Ser.*, 31(4), 1043–1048.

Liu, B.Z. and Deng, G.M. (1991). An improved mathematical model of hormone secretion in the hypothalamo-pituitary-gonadal axis in man. *J. Theor. Biol.*, 150, 51–58.

Luenberger, D.G. (1979). *Introduction to Dynamic Systems: Theory, Models, and Applications.* Wiley, New York.

Mackey, M.C., Santillán, M., Tyran-Kamińska, M., and Zeron, E.S. (2012–2014). The utility of simple mathematical models in understanding gene regulatory dynamics. *In Silico Biol.*, 12(1–2), 23–53.

Mukhopadhyay, B. and Bhattacharyya, R. (2004). A delayed mathematical model for testosterone secretion with feedback control mechanism. *Int. J. Math. Math. Sci.*, 2004(3), 105–115.

Murray, J.D. (2002). *Mathematical Biology, I: An Introduction (3rd ed.).* Springer, New York.

Ren, H. (2004). Stability analysis of a simplified model for the control of testosterone secretion. *Discrete Contin. Dynam. Syst., Ser. B*, 4(3), 729–738.

Samoilenko, A.M. and Perestyuk, N.A. (1995). *Impulsive Differential Equations.* World Scientific, Singapore.

Smith, W.R. (1980). Hypothalamic regulation of pituitary secretion of lutheinizing hormone—II Feedback control of gonadotropin secretion. *Bull. Math. Biol.*, 42, 57–78.

Smith, W.R. (1983). Qualitative mathematical models of endocrine systems. *Am. J. Physiol.*, 245(4), R473–R477.

Stamova, I. and Stamov, G. (2016). *Applied Impulsive Mathematical Models.* Springer, Berlin.

Sun, X., Yuan, R., and Cao, J. (2016). Bifurcations for Goodwin model with three delays. *Nonlinear Dynam.*, 84(2), 1093–1105.

Zhang, Y., Cao, J., and Xu, W. (2015). Stability and Hopf bifurcation of a Goodwin model with four different delays. *Neurocomputing*, 165(C), 144–151.

Zhusubaliyev, Zh.T., Mosekilde, E., Churilov, A.N., and Medvedev, A. (2015). Multistability and hidden attractors in an impulsive Goodwin oscillator with time delay. *Eur. Phys. J. Special Topics*, 224, 1519–1539.

Zhusubaliyev, Z., Churilov, A., and Medvedev, A. (2014). Time delay induced multistability and complex dynamics in an impulsive model of endocrine regulation. In *Proc. 13th European Control Conf. (ECC)*, 2304–2309. Strasbourg, France.

Appendix A. PROOF OF THEOREM 4

Formulas (7) yield

$$A_1 = (D_0 - A_0)\mathrm{e}^{A_0\tau_1}, \quad A_2 = (D - D_0)\mathrm{e}^{D_0\tau_2},$$

so (10) can be rewritten as

$$\begin{aligned} x' &= A_0 x(t) + (D_0 - A_0)\mathrm{e}^{A_0\tau_1} x(t-\tau_1) \\ &\quad + (D - D_0)\mathrm{e}^{D_0\tau_2} x(t-\tau_2). \end{aligned}$$

With $D = A_0 + (D_0 - A_0) + (D - D_0)$, the last equality is equivalent to

$$\begin{aligned} x' &= Dx(t) - (D_0 - A_0)\left[x(t) - \mathrm{e}^{A_0\tau_1}x(t-\tau_1)\right] \\ &\quad - (D - D_0)\left[x(t) - \mathrm{e}^{D_0\tau_2}x(t-\tau_2)\right]. \end{aligned} \tag{A.1}$$

From the previously deduced formulas

$$D_0 - A_0 = \begin{bmatrix} 0 & 0\ 0 \\ G_1\mathrm{e}^{-U_1\tau_1} & 0\ 0 \\ 0 & 0\ 0 \end{bmatrix},$$

$$D - D_0 = \begin{bmatrix} 0 & 0 \\ G_z & 0 \end{bmatrix},$$

i. e. the last columns of these matrices are zero. Using (8), equation (A.1) can be expressed as

$$x' = Dx(t) - (D_0 - A_0)\eta(t) - (D - D_0)\xi(t), \tag{A.2}$$

where

$$\eta(t) = \begin{bmatrix} \eta_u(t) \\ 0 \end{bmatrix}, \quad \xi(t) = \begin{bmatrix} \xi_z(t) \\ 0 \end{bmatrix} \tag{A.3}$$

and

$$\eta_u(t) = u(t) - \mathrm{e}^{U_1\tau_1}u(t-\tau_1), \tag{A.4}$$

$$\xi_z(t) = z(t) - \mathrm{e}^{D_z\tau_2}z(t-\tau_2). \tag{A.5}$$

Lemma 5. Let $n \geqslant 1$. Then from the first formula (A.3) we obtain

$$\eta(t) = \alpha(t), \quad t > t_1. \tag{A.6}$$

Moreover, $\eta(t) = 0$ for $t_0^* < t < t_1$.

Proof. Since

$$u' = U_1 u(t), \qquad t \neq t_n,$$

it follows that

$$u(t) = \mathrm{e}^{U_1(t-t_n)} \times \begin{cases} u(t_n^-), & t_{n-1} < t < t_n, \\ u(t_n^+), & t_n < t < t_{n+1}. \end{cases} \tag{A.7}$$

Then

$$u(t-\tau_1) = \mathrm{e}^{U_1(t-t_n^*)} \times \begin{cases} u(t_n^-), & t_{n-1}^* < t < t_n^*, \\ u(t_n^+), & t_n^* < t < t_{n+1}^*. \end{cases} \tag{A.8}$$

From (A.4), (A.7), (A.8) we get

$$\eta_u(t) = \mathrm{e}^{U_1(t-t_n)} \times \begin{cases} 0, & t_{n-1}^* < t < t_n, \\ u(t_n^+) - u(t_n^-), & t_n < t < t_n^*, \\ 0, & t_n^* < t < t_{n+1}. \end{cases} \tag{A.9}$$

Then (16), (A.9) imply

$$\eta_u(t) = \begin{cases} 0, & t_{n-1}^* < t < t_n, \\ \lambda_n \mathrm{e}^{U_1(t-t_n)}B_u, & t_n < t < t_n^*, \\ 0, & t_n^* < t < t_{n+1}. \end{cases} \tag{A.10}$$

Since

$$\mathrm{e}^{A_0 t}B = \begin{bmatrix} \mathrm{e}^{U_1 t}B_u \\ 0 \end{bmatrix}$$

for all t, the statement of Lemma 5 follows from (A.10) and (23). □

Deduce explicit formulas for the function $z(t)$. Consider a matrix A_z and a vector B_z defined as

$$A_z = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}, \quad B_z = \begin{bmatrix} B_u \\ 0 \end{bmatrix}.$$

(The number of elements in B_z is $p_1 + p_2$.) Recall that the matrix D_z is defined by (8).

Lemma 6. The function $z(t)$ can be calculated as

$$z(t) = \mathrm{e}^{D_z(t-t_n)} z(t_n^-) + \lambda_n \theta_z(t) B_z, \tag{A.11}$$

where

$$\theta_z(t) = \begin{cases} 0, & t_{n-1}^* < t < t_n, \\ \mathrm{e}^{A_z(t-t_n)}, & t_n < t < t_n^*, \\ \mathrm{e}^{D_z(t-t_n^*)} \mathrm{e}^{A_z \tau_1}, & t_n^* < t < t_{n+1}. \end{cases} \tag{A.12}$$

Proof. Notice that $w(t)$, $\xi(t)$ are not involved in the first two block rows of (A.2). We have

$$D_z - A_z = \begin{bmatrix} 0 & 0 \\ G_1 \mathrm{e}^{-U_1 \tau_1} & 0 \end{bmatrix}.$$

Then the first p_1+p_2 rows of equation (A.2) can be written as

$$z' = D_z z(t) - (D_z - A_z) \eta_z(t), \quad \eta_z(t) = \begin{bmatrix} \eta_u(t) \\ 0 \end{bmatrix}, \tag{A.13}$$

where 0 stands for the zero vector of dimension p_2. Since $\eta_u(t)$ is defined by (A.10) and

$$\mathrm{e}^{A_z t} B_z = \begin{bmatrix} \mathrm{e}^{U_1 t} B_u \\ 0 \end{bmatrix} \quad \text{for all} \quad t,$$

the function $\eta_z(t)$ in (A.13) takes the form of

$$\eta_z(t) = \begin{cases} 0, & t_{n-1}^* < t < t_n, \\ \lambda_n \mathrm{e}^{A_z(t-t_n)} B_z, & t_n < t < t_n^*, \\ 0, & t_n^* < t < t_{n+1}. \end{cases} \tag{A.14}$$

If $t_{n-1}^* < t < t_n$ then (A.13), (A.14) imply $z' = D_z z(t)$. Then the first case in (A.11) follows.

Let us consider the case $t_n < t < t_{n+1}$. In this interval the function $z(t)$ is continuous, while $\eta_z(t)$ has a jump at $t = t_n^*$. Evidently,

$$\eta_z' = A_z \eta_z(t), \quad t_n < t < t_{n+1},\ t \neq t_n^*.$$

Consider a difference $y_z(t) = z(t) - \eta_z(t)$. Then

$$\begin{aligned} y_z' &= D_z y_z(t), \quad t_n < t < t_{n+1},\ t \neq t_n^*, \\ y_z(t_n^+) &= z(t_n^+) - \lambda_n B_z = z(t_n^-), \\ y_z(t_n^+ + \tau_1) - y_z(t_n^- + \tau_1) &= \eta_z(t_n^- + \tau_1) - \eta_z(t_n^+ + \tau_1) \\ &= \lambda_n \mathrm{e}^{A_z \tau_1} B_z. \end{aligned} \tag{A.15}$$

Relationships (A.15) imply

$$y_z(t) = \mathrm{e}^{D_z(t-t_n)} z(t_n^-) + \begin{cases} 0, & t_n < t < t_n^*, \\ \lambda_n \mathrm{e}^{D_z(t-t_n^*)} \mathrm{e}^{A_z \tau_1} B_z, & t_n^* < t < t_{n+1}. \end{cases} \tag{A.16}$$

Since $z(t) = y_z(t) + \eta_z(t)$, from (A.14), (A.16) we come to (A.11) for $t_n < t < t_{n+1}$. $\square$

Consider now the calculation of $\xi(t)$. Recall that the functions $\xi(t)$, $\xi_z(t)$ are defined by (A.3), (A.5).

Lemma 7. Let $n \geqslant 1$. The function $\xi_z(t)$ defined by (A.5) is calculated as

$$\xi_z(t) = \lambda_n \tilde{\theta}_z(t) B_z, \quad t_n < t < t_{n+1}, \tag{A.17}$$

where

$$\tilde{\theta}_z(t) = \begin{cases} \mathrm{e}^{A_z(t-t_n)}, \quad t_n < t < t_n^*, \\ \mathrm{e}^{D_z(t-t_n^*)} \mathrm{e}^{A_z \tau_1}, \quad t_n^* < t < \hat{t}_n, \\ \mathrm{e}^{D_z(t-t_n^*)} \mathrm{e}^{A_z \tau_1} - \mathrm{e}^{D_z \tau_2} \mathrm{e}^{A_z(t-\hat{t}_n)}, \\ \qquad \hat{t}_n < t < t_n^{**}, \\ 0, \quad t_n^{**} < t < t_{n+1}. \end{cases} \tag{A.18}$$

Proof. From (A.11) it follows that

$$z(t-\tau_2) = \mathrm{e}^{D_z(t-\hat{t}_n)} z(t_n^-) + \lambda_n \theta_z(t-\tau_2) B_z, \tag{A.19}$$

$t_n < t < t_{n+1}$. Hence from (A.5) we have

$$\xi_z(t) = \lambda_n \big(\theta_z(t) - \mathrm{e}^{D_z \tau_2} \theta_z(t-\tau_2)\big) B_z, \tag{A.20}$$

$t_n < t < t_{n+1}$. Thus (A.20) can be represented as (A.17) with

$$\tilde{\theta}_z(t) = \theta_z(t) - \mathrm{e}^{D_z \tau_2} \theta_z(t-\tau_2).$$

Formula (A.12) implies

$$\theta_z(t-\tau_2) = \begin{cases} 0, & t_n < t < \hat{t}_n, \\ \mathrm{e}^{A_z(t-\hat{t}_n)}, & \hat{t}_n < t < t_n^{**}, \\ \mathrm{e}^{D_z(t-t_n^{**})} \mathrm{e}^{A_z \tau_1}, & t_n^{**} < t < t_{n+1}. \end{cases}$$

Thus we obtain (A.18).

Lemma 8. Let $n \geqslant 1$. Then

$$G_z \xi_z(t) = G_z \tilde{\xi}_z(t), \quad t_n < t < t_{n+1}, \tag{A.21}$$

where

$$\tilde{\xi}_z(t) = \begin{cases} \lambda_n \mathrm{e}^{A_z(t-t_n)} B_z, & t_n < t < t_n^*, \\ \lambda_n \mathrm{e}^{D_z(t-t_n^*)} \mathrm{e}^{A_z \tau_1} B_z, & t_n^* < t < t_n^{**}, \\ 0, & t_n^{**} < t < t_{n+1}. \end{cases} \tag{A.22}$$

Proof. From (A.18) and (A.22) we have

$$\xi_z(t) - \tilde{\xi}_z(t) = \begin{cases} -\lambda_n \mathrm{e}^{D_z \tau_2} \mathrm{e}^{A_z(t-\hat{t}_n)} B_z, & \hat{t}_n < t < t_n^{**}, \\ 0, & \text{otherwise}. \end{cases} \tag{A.23}$$

Since

$$G_z \mathrm{e}^{D_z \tau_2} \mathrm{e}^{A_z t} B_z = [0 \quad G_2] \begin{bmatrix} \mathrm{e}^{U_1 t} & 0 \\ 0 & \mathrm{e}^{U_2 t} \end{bmatrix} \begin{bmatrix} B_u \\ 0 \end{bmatrix} = 0$$

for all t, (A.23) implies (A.21). $\square$

Now we can complete the proof of Theorem 4. From (A.21) we have

$$(D - D_0)\xi(t) = \begin{bmatrix} 0 & 0 \\ G_z & 0 \end{bmatrix} \begin{bmatrix} \xi_z(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ G_z & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_z(t) \\ 0 \end{bmatrix}.$$

At the same time

$$\mathrm{e}^{A_0 t} B = \begin{bmatrix} \mathrm{e}^{A_z t} B_z \\ 0 \end{bmatrix}, \quad \mathrm{e}^{D_0 t} \mathrm{e}^{A_0 \tau_2} B = \begin{bmatrix} \mathrm{e}^{D_z t} \mathrm{e}^{A_z \tau_2} B_z \\ 0 \end{bmatrix}$$

for all t. Thus from (A.22) we have

$$(D - D_0)\xi(t) = (D - D_0)(\alpha(t) + \beta(t))$$

and for $n \geqslant 1$, $t > t_1$ any solution $(t_n, x(t))$ satisfies

$$x' = Dx(t) - (D_0 - A_0)\alpha(t) - (D - D_0)(\alpha(t) + \beta(t)). \tag{A.24}$$

Let $t_n < t < t_n^*$. Then $\beta(t) \equiv 0$. Since

$$(D_0 - A_0) + (D - D_0) = D - A_0,$$

from (A.24) we conclude that

$$x' = Dx(t) - (D - A_0)\alpha(t), \quad t_n < t < t_n^*.$$

Let $t_n^* < t < t_n^{**}$ then $\alpha(t) \equiv 0$, so (A.24) can be rewritten as

$$x' = Dx(t) - (D - D_0)\beta(t), \quad t_n^* < t < t_n^{**}.$$

Finally, let $t_n^{**} < t < t_{n+1}$. Then the functions $\alpha(t)$, $\beta(t)$ are equal to zero, so we come to (22) again. The proof of Theorem 4 is complete.