Necessary and Sufficient Conditions for Consensus Tracking of Multi-Agent Time-Delay Systems

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Abstract: In this paper, a distributed protocol is proposed to solve the consensus tracking problem under heterogeneous input and communication delays. In contrast to consensus which can be achieved even without knowledge of the communication delay, tracking a general trajectory requires precise information about the individual delays. The proposed protocol is customized as a tracking controller in conjunction with a consensus-based estimator for the desired trajectory. The tracking controllers accommodate the input delays whereas communication delays govern the stability of the estimators. The tracking problem of single-integrator agents is first addressed and later adapted for double-integrator agents. The choice of coupled single-integrator estimators for the double-integrator agents eases gain tuning. Simulations are carried out to demonstrate the effectiveness of the proposed technique.

Keywords: Trajectory Tracking, Delay, Root Tendency, Multi-Agent System.

1. INTRODUCTION

Cooperative control of multi-agent system has attracted a lot of attention in the past decade. The study of multi-agent cooperative control include synchronization, tracking, formation, flocking. Several consensus algorithms have been used to solve these problems and can be found in Panteley and Loria (2015); Hu (2012); Cao et al. (2013); Tanner et al. (2007) and references therein. Consensus algorithms are distributive in nature and consider exchange of information among neighbouring agents such that all agents in the network agree on a common value (Ren et al., 2007; Chen et al., 2011; Qin et al., 2012). In the present work, we mainly focus on consensus tracking which means the whole group follows autonomously a time-varying reference trajectory. In particular, we address the effect of both communication and input delays on the stability of our proposed consensus tracking protocol.

The problem of consensus tracking has been addressed in Hu (2012); Peng et al. (2013); Hu et al. (2015), where the effect of delay is not considered. In multi-agent system, instantaneous exchange of information among neighbouring agents may not be possible leading to communication delay. Delays in the actuation of an agent, (dos Santos Junior et al., 2015) are referred as input delays. In the existing literature, two types of consensus algorithms can be found for accommodating communication delay. The first one deals with the relative difference of current state of the concerned agent and the delayed state of its neighbouring agent (Seuret et al., 2008). The second approach considers the relative difference of delayed state of both the concerned agent and its neighbouring agent (Olfati-Saber and Murray, 2004; Zhang et al., 2017). In our work, we consider the presence of communication delay and follow the second approach. Consensus of multi-agent system with both communication and input delays has been studied in Tian and Liu (2008). The authors show that consensus conditions do not depend on communication delay. However, for the tracking problem, consensus conditions may depend both on the communication and input delay. Tian and Liu (2009) consider diverse input delays and solve the tracking problem for double-integrator agents when reference trajectory has a constant velocity. Xie and Cheng (2014) derive conditions for trajectory tracking for double-integrator agents in the presence of homogeneous communication delays. The work in Meng et al. (2011) addresses tracking issues when both type of delays are present and find conditions for uniform ultimate boundedness of the tracking errors. To the best of our knowledge, asymptotic convergence of the tracking error to zero in the presence of heterogeneous communication and input delays has not been reported yet. In De et al. (2017), a consensus-based estimator with a tracking controller has been designed to track a time-varying trajectory in the presence of heterogeneous input delays with no communication delay. In this work, we extend the control architecture reported in De et al. (2017) to accommodate heterogeneous communication delays. The major contributions of this paper can be summarized as

- Necessary and sufficient conditions are derived for tracking a time-varying trajectory in the presence of heterogeneous communication and input delays for the tracking controller coupled with a consensus-based estimator.

- It is found that controller and estimator gains are inversely proportional to the input delays and communication delays respectively. Arbitrarily large bounded communication and input delays can be tolerated by suitably adjusting the gains.
- Modelling the estimator as coupled single-integrator agents helps gain tuning for double-integrator agents. This benefit can be extended to higher order integrators.

2. PROBLEM FORMULATION

The objective of a group of autonomous agents is to track a desired trajectory in spite of heterogeneous communication and input delays. We first design a protocol for a group of single-integrator agents, and then, extend this idea to double-integrator agents. In our design agent i pursues agent $i+1$ if it has no information on the desired trajectory, $x_d(t)$. However, when the information on desired trajectory is available to i^{th} agent, it pursues the desired trajectory.

Consider a group of N agents with single-integrator dynamics. The dynamics of i^{th} agent, $i = 1, 2, \ldots, N$ is

$$\dot{x}_i(t) = u_i\left(t - \tau_i^{in}\right) \tag{1}$$

where $x_i \in \mathbb{R}$ is the position of i^{th} agent, and $u_i \in \mathbb{R}$ is the control input. The control input for i^{th} agent is delayed by τ_i^{in}. The input delay τ_i^{in} is heterogeneous through out the network. We also consider the presence of heterogeneous communication delay τ_i^{com} for $i = 1, 2, \ldots, N$ when information flows from $(i+1)^{\text{th}}$ agent to i^{th} agent over communication channel.

Assumption 1. Both the communication delays τ_i^{com} and input delays τ_i^{in}, $\forall i$ are constant and bounded.

We aim to design a protocol such that for any initial position $x_i(0)$, all agents converge to the desired trajectory, that is,

$$\lim_{t\to\infty} \|x_d(t) - x_i(t)\| = 0 \tag{2}$$

in the presence of heterogeneous communication delays τ_i^{com} and input delays τ_i^{in}. We make the following assumption along with Assumption 1:

Assumption 2. For single-integrator agents, only $\dot{x}_d(t)$ is available to all agents for all time.

Next, we attempt to solve the tracking problem when the agents have double-integrator dynamics. The dynamics of i^{th} agent, $i = 1, 2, \ldots, N$ is given by

$$\left.\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i\left(t - \tau_i^{in}\right), \end{aligned}\right\} \tag{3}$$

where $x_i \in \mathbb{R}$ and $v_i \in \mathbb{R}$ are the position and velocity of i^{th} agent, and $u_i \in \mathbb{R}$ is the control input. We make the following assumption along with Assumption 1:

Assumption 3. For double-integrator agents, only $\ddot{x}_d(t)$ is available to all agents for all time.

In this scenario, we aim to design a protocol such that for any initial position $x_i(0)$ and velocity $v_i(0)$, all agents converge to the desired trajectory, that is $\forall i$,

$$\left.\begin{aligned} \lim_{t\to\infty} \|x_d(t) - x_i(t)\| = 0, \\ \lim_{t\to\infty} \|\dot{x}_d(t) - v_i(t)\| = 0, \end{aligned}\right\} \tag{4}$$

in the presence of heterogeneous communication delays τ_i^{com} and input delays τ_i^{in}.

3. CONSENSUS TRACKING WITH TIME-DELAYS

To track a general desired trajectory we present a consensus-based estimator along with a tracking controller for each agent. The purpose of the estimator is to estimate the desired trajectory whereas the tracking controller is responsible to reduce the error between estimates and actual states of the agent. When one agent has information on the desired trajectory it updates its estimator based on the available information. In the absence of desired trajectory information, the i^{th} agent updates its estimate using the information received from agent $(i+1)$.

3.1 Single-integrator model

The control law for single-integrator agents is designed as

$$u_i(t) = \dot{x}_d\left(t + \tau_i^{in}\right) - K_i\left(x_i(t) - \hat{x}_i(t)\right),\ i = 1, 2, \ldots, N, \tag{5}$$

where $\hat{x}_i(t)$ represents estimate of desired trajectory made by i^{th} agent and $K_i \in \mathbb{R}$ is the respective controller gain. The estimator has the dynamics

$$\begin{aligned} \dot{\hat{x}}_i(t) = &\dot{x}_d(t) + c_i\left(1 - a_{id}\right)\left(\hat{x}_{i+1}\left(t - \tau_i^{com}\right)\right. \\ &\left.-\hat{x}_i\left(t - \tau_i^{com}\right)\right) + \gamma_i a_{id}\left(x_d(t) - \hat{x}_i(t)\right), \end{aligned} \tag{6}$$

where $c_i, \gamma_i \in \mathbb{R}$. The weight a_{id} is 1 if i^{th} agent has the knowledge of $x_d(t)$, and 0 otherwise. Before deriving the conditions on the design parameters, we present a lemma which is essential to prove the necessary and sufficient conditions for tracking.

This lemma extends the condition for the roots of the quasipolynomial $\rho_1(s) = s + ae^{-s\tau}$ to be in the open left-half complex plane presented in De et al. (2017).

Lemma 1. The characteristic quasipolynomial $\rho_1(s) = s + ae^{-s\tau}$ has all the roots in the open left-half complex plane if and only if $0 < a < \dfrac{\pi}{2\tau}$.

Proof. We will prove this by the method of contradiction. If $a = 0$, $\rho_1(s)$ will always has a root at the origin. Now let a be negative. When $\tau = 0$, the root of $\rho_1(s)$ is at $s = -a > 0$. Therefore, the root is in the open right-half complex plane. Let us analyze the behavior of the roots as τ increases. For all the roots of $\rho_1(s)$ to have negative real parts for some $\tau (> 0)$, roots in the open right-half plane have to cross the imaginary axis from right-half to left-half complex plane. As $s = 0$ cannot be a root for $a < 0$, the roots will have to cross the imaginary axis at $j\omega$, $\omega > 0$. We get the root sensitivity $\left.\dfrac{ds}{d\tau}\right|_{s=j\omega}$ as

$$\begin{aligned} \left.\frac{ds}{d\tau}\right|_{s=j\omega} &= -\left.\frac{\frac{\partial \rho_1(s)}{\partial \tau}}{\frac{\partial \rho_1(s)}{\partial s}}\right|_{s=j\omega} = \left.\frac{ase^{-s\tau}}{1 - a\tau e^{-s\tau}}\right|_{s=j\omega} \\ &= \left.\frac{-s^2}{1 + s\tau}\right|_{s=j\omega} = \frac{\omega^2}{1 + j\omega\tau} = \frac{\omega^2(1 - j\omega\tau)}{1 + \omega^2\tau^2}. \end{aligned}$$

Hence, the root tendency is

$$\begin{aligned}\mathrm{RT}|_{s=j\omega} &= \mathrm{sgn}\left[Re\left(\left.\frac{ds}{d\tau}\right|_{s=j\omega}\right)\right]\\ &= \mathrm{sgn}\left[\frac{\omega^2}{1+\omega^2\tau^2}\right] = +1. \end{aligned} \tag{7}$$

This signifies that if at least one root is in the open right-half complex plane, then that particular root cannot cross the imaginary axis from right to left, and hence, for $a < 0$ there always will be at least one root in the open right-half complex plane for any delay $0 \le \tau < \infty$. When the parameter $a > 0$, then following De et al. (2017) the delay margin can be found as $\tau = \frac{\pi}{2a}$. Therefore, all the roots of $\rho_1(s)$ will be in the open left-half complex plane if and only if $0 < a < \dfrac{\pi}{2\tau}$. ∎

Lemma 1 is useful for designing the parameters K_i and c_i. The necessary and sufficient conditions for tracking a desired trajectory are presented in Theorem 1.

Theorem 1. Consider a group of Single-integrator agents given by (1). The control law (5) along with the estimator (6) solves the tracking problem if and only if

$$\begin{aligned}0 < K_i < \frac{\pi}{2\tau_i^{in}}, \quad \forall i,\\ \left.\begin{aligned}0 < c_i < \frac{\pi}{2\tau_i^{com}}, &\quad \text{if} \quad a_{id}=0,\\ \gamma_i > 0, &\quad \text{if} \quad a_{id}=1\end{aligned}\right\}, \quad \forall i.\end{aligned}$$

Proof. We define two error variables as $\epsilon_{i1}(t) = x_d(t) - \hat{x}_i(t)$, and $\epsilon_{i2}(t) = x_i(t) - \hat{x}_i(t)$. The tracking problem can be seen as the convergence of the error variables $\epsilon_1(t)$ and $\epsilon_2(t)$ to zero with $\epsilon_1 = [\epsilon_{11}\,\epsilon_{21}\ldots\epsilon_{N1}]^\top$, and $\epsilon_2 = [\epsilon_{12}\,\epsilon_{22}\ldots\epsilon_{N2}]^\top$. Using (5) and (6), time derivative of $\epsilon_{i1}(t)$ and $\epsilon_{i2}(t)$ can be found as

$$\begin{aligned}\dot{\epsilon}_{i1}(t) =&\dot{x}_d(t) - \dot{\hat{x}}_i(t)\\ =&- c_i\left(1-a_{id}\right)\left(\epsilon_{i1}\left(t-\tau_i^{com}\right)\right.\\ &\left.-\epsilon_{(i+1)1}\left(t-\tau_i^{com}\right)\right) - \gamma_i a_{id}\epsilon_{i1}(t),\end{aligned} \tag{8}$$

$$\begin{aligned}\dot{\epsilon}_{i2}(t) =&\dot{x}_i(t) - \dot{\hat{x}}_i(t)\\ =&- K_i\epsilon_{i2}\left(t-\tau_i^{in}\right) - c_i\left(1-a_{id}\right)\\ &\times\left(\epsilon_{i1}\left(t-\tau_i^{com}\right) - \epsilon_{(i+1)1}\left(t-\tau_i^{com}\right)\right) - \gamma_i a_{id}\epsilon_{i1}(t).\end{aligned} \tag{9}$$

The dynamics of the errors can be represented as

$$\begin{aligned}\dot{\epsilon}(t) =& \begin{bmatrix}\Gamma & 0\\ \Gamma & 0\end{bmatrix}\epsilon(t) + \sum_{i=1}^{N}\left\{\begin{bmatrix}C_i & 0\\ C_i & 0\end{bmatrix}\right.\\ &\left.\times\,\epsilon\left(t-\tau_i^{com}\right)\right\} + \sum_{i=1}^{N}\left\{\begin{bmatrix}0 & 0\\ 0 & A_i\end{bmatrix}\epsilon\left(t-\tau_i^{in}\right)\right\},\end{aligned} \tag{10}$$

where $\epsilon(t) = \left[\epsilon_1^\top(t)\ \epsilon_2^\top(t)\right]^\top$. The matrices Γ, C_i, and A_i are defined in (11).

$$\Gamma = \begin{bmatrix}-\gamma_1 a_{1d} & & & & \\ & \ddots & & & \\ & & -\gamma_i a_{id} & & \\ & & & \ddots & \\ & & & & -\gamma_N a_{Nd}\end{bmatrix}, \quad C_i = \begin{bmatrix}0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & -c_i\left(1-a_{id}\right) & c_i\left(1-a_{id}\right) & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}, \quad A_i = \begin{bmatrix}0 & & & & \\ & \ddots & & & \\ & & -K_i & & \\ & & & \ddots & \\ & & & & 0\end{bmatrix}. \tag{11}$$

$$\Delta(s) = \det\left(\begin{bmatrix}sI_N - \Gamma - \sum_{i=1}^{N} C_i e^{-s\tau_i^{com}} & 0\\ -\Gamma - \sum_{i=1}^{N} C_i e^{-s\tau_i^{com}} & sI_N - \sum_{i=1}^{N} A_i e^{-s\tau_i^{in}}\end{bmatrix}\right) = \det\left(sI_N - \Gamma - \sum_{i=1}^{N} C_i e^{-s\tau_i^{com}}\right) \times \det\left(sI_N - \sum_{i=1}^{N} A_i e^{-s\tau_i^{in}}\right) \tag{12}$$

The characteristic quasipolynomial of (10) is conferred in (12) and can be interpreted as

$$\Delta(s) = \rho_a(s).\rho_b(s), \tag{13}$$

with $\rho_a(s) = \prod_{i=1}^{N}\left(s + \gamma_i a_{id} + c_i\left(1-a_{id}\right)e^{-s\tau_i^{com}}\right)$, $\rho_b(s) = \prod_{i=1}^{N}\left(s + K_i e^{-s\tau_i^{in}}\right)$. The error $\epsilon(t)$ decays to zero if and only if all the roots of (13) are in the open left-half complex plane. Following Lemma 1, the necessary and sufficient conditions for the roots of $\rho_b(s)$ to be in the open left-half complex plane can be obtained as

$$0 < K_i < \frac{\pi}{2\tau_i^{com}}, \forall i. \tag{14}$$

For $a_{id} = 1$, the root of $\left(s + \gamma_i a_{id} + c_i\left(1-a_{id}\right)e^{-s\tau_i^{com}}\right)$ is at $s = -\gamma_i$. This establishes that for stability $\gamma_i > 0$, when $a_{id} = 1$. When $a_{id} = 0$, we require roots of $\left(s + c_i e^{-s\tau_i^{com}}\right)$ to have negative real parts. With the aid of Lemma 1, we get condition on c_i as $0 < c_i < \frac{\pi}{2\tau_i^{com}}$. Therefore, roots of $\rho_a(s)$ are in the open left-half complex plane if and only if

$$\left.\begin{aligned}0 < c_i < \frac{\pi}{2\tau_i^{com}}, &\quad \text{if} \quad a_{id}=0,\\ \gamma_i > 0, &\quad \text{if} \quad a_{id}=1\end{aligned}\right\} \quad \forall i. \tag{15}$$

The error $\epsilon(t)$ decays to zero if and only if the design parameters satisfy (14) and (15). As $\epsilon(t) \to 0$, all agents track the desired trajectory $x_d(t)$. ∎

Remark 1. Controller gains K_i and estimator gains c_i can be designed independently. Controller and estimator gains are inversely proportional to τ_i^{in} and τ_i^{com} respectively.

Next, the objective is to develop a tracking algorithm for double-integrator case. In view of this, the tracking controller has been designed along with an estimator that captures the structure of the estimator used in single-integrator case. The estimator dynamics in this case resembles two coupled single-integrator estimators.

3.2 Double-integrator model

The control law for the double-integrator agents is designed as

$$u_i(t) = \ddot{x}_d\left(t+\tau_i^{in}\right) - K_{ip}\left(x_i(t) - \hat{x}_i(t)\right) - K_{iv}\left(v_i(t) - \hat{v}_i(t)\right), \quad i = 1, 2, \dots, N, \tag{16}$$

where $\hat{x}_i(t)$ and $\hat{v}_i(t)$ represent estimated position and velocity of i^{th} agent. The gains $K_{ip}, K_{iv} \in \mathbb{R}$ are the design parameters. The estimator has the dynamics

$$\left.\begin{aligned} \dot{\hat{x}}_i(t) =& \hat{v}_i(t) + c_{ip}\left(1-a_{id}\right)\left(\hat{x}_{i+1}\left(t-\tau_i^{com}\right)\right. \\ & \left.-\hat{x}_i\left(t-\tau_i^{com}\right)\right) + \gamma_{ip} a_{id}\left(x_d(t) - \hat{x}_i(t)\right) \\ \dot{\hat{v}}_i(t) =& \ddot{x}_d(t) + c_{iv}\left(1-a_{id}\right)\left(\hat{v}_{i+1}\left(t-\tau_i^{com}\right)\right. \\ & \left.-\hat{v}_i\left(t-\tau_i^{com}\right)\right) + \gamma_{iv} a_{id}\left(\dot{x}_d(t) - \hat{v}_i(t)\right), \end{aligned}\right\} \tag{17}$$

where $c_{ip}, c_{iv}, \gamma_{ip}, \gamma_{iv} \in \mathbb{R}$. Without loss of generality, we can take $K_{ip} = \frac{\beta_i}{2} K_{iv}^2$ with $\beta_i \in \mathbb{R}$. Here, we present a lemma that analyzes the characteristics of roots of $\rho_2(s) = s^2 + bse^{-s\tau} + ce^{-s\tau}$. This is required to prove the necessity and sufficiency of the obtained conditions.

Lemma 2. The characteristic quasipolynomial $\rho_2(s) = s^2 + bse^{-s\tau} + ce^{-s\tau}$ has all the roots in the open left half complex plane if and only if $b, c > 0$, and $\tau < \frac{1}{\sqrt{\frac{b^2+\sqrt{b^4+4c^2}}{2}}} \tan^{-1} \sqrt{\frac{b^2+\sqrt{b^4+4c^2}}{2c^2}}\, b.$

Proof. Firstly for $c = 0$, $\rho_2(s)$ will have one root at origin for any τ. Similar to Lemma 1, we will prove by contradiction. When $\tau = 0$, the roots are at $s = \frac{-b \pm \sqrt{b^2-4c}}{2}$. It can be noted that there will be at least one root in the closed right-half complex plane when at least one of the coefficients of $\rho_2(s)$ is nonpositive. The root sensitivity at $j\omega$ crossing can be found as

$$\begin{aligned} \left.\frac{ds}{d\tau}\right|_{s=j\omega} &= \left.\frac{bs^2 e^{-s\tau} + cse^{-s\tau}}{2s + b\left(-\tau s e^{-s\tau} + e^{-s\tau}\right) - c\tau e^{-s\tau}}\right|_{s=j\omega} \\ &= \frac{\omega^2(c + j\omega b)}{-\omega^2 b\tau + 2c + j\omega(b + c\tau)} \\ &= \frac{\omega^2(c + j\omega b)\left((2c - \omega^2 b\tau) - j\omega(b + c\tau)\right)}{(2c - \omega^2 b\tau)^2 + \omega^2(b + c\tau)^2} \end{aligned}$$

Hence, the root tendency is

$$\begin{aligned} \mathrm{RT}|_{s=j\omega} &= \operatorname{sgn}\left[Re\left(\left.\frac{ds}{d\tau}\right|_{s=j\omega}\right)\right] \\ &= \operatorname{sgn}\left[\frac{\omega^2\left(2c^2 + \omega^2 b^2\right)}{(2c - \omega^2 b\tau)^2 + \omega^2(b + c\tau)^2}\right] = +1. \end{aligned} \tag{18}$$

Presence of at least one root in the closed right-half complex plane at $\tau = 0$ and the root tendency given by (18) reveals that the $\rho_2(s)$ has at least one root in the closed right-half complex plane always. This contradicts what we assume. Therefore, the necessary condition for stability is that $b, c > 0$. Under this scenario, following De et al. (2017) the condition for all roots to be in the open left-half complex plane can be found as

$$\tau < \frac{1}{\sqrt{\frac{b^2+\sqrt{b^4+4c^2}}{2}}} \tan^{-1} \sqrt{\frac{b^2+\sqrt{b^4+4c^2}}{2c^2}}\, b.$$

■

Lemma 1 and 2 are useful to derive the necessary and sufficient conditions for asymptotic convergence of our control protocol for double-integrator case. The conditions on the design parameters are presented in Theorem 2.

Theorem 2. Consider a group of double-integrator agents given by (3). The control law (16) along with the estimator (17) solves the tracking problem if and only if

$$0 < K_{iv} < \frac{\tan^{-1}\left(\frac{2}{\beta_i}\sqrt{\frac{1+\sqrt{1+\beta_i^2}}{2}}\right)}{\tau_i^{in}\sqrt{\frac{1+\sqrt{1+\beta_i^2}}{2}}},\ \beta_i > 0,\ \forall i,$$

$$\left.\begin{aligned} 0 < c_{ip}, c_{iv} < \frac{\pi}{2\tau_i^{com}}, \quad & \text{if} \quad a_{id} = 0, \\ \gamma_{ip}, \gamma_{iv} > 0, \quad & \text{if} \quad a_{id} = 1 \end{aligned}\right\},\ \forall i.$$

Proof. We define error variables as $\delta_{i1}(t) = x_d(t) - \hat{x}_i(t)$, $\delta_{i2}(t) = \dot{x}_d(t) - \hat{v}_i(t)$, $\delta_{i3}(t) = x_i(t) - \hat{x}_i(t)$, $\delta_{i4}(t) = v_i(t) - \hat{v}_i(t)$. Also, we define $\delta_1 = [\delta_{11}\ \delta_{21} \dots \delta_{N1}]^\top$, $\delta_2 = [\delta_{12}\ \delta_{22} \dots \delta_{N2}]^\top$, $\delta_3 = [\delta_{13}\ \delta_{23} \dots \delta_{N3}]^\top$ and $\delta_4 = [\delta_{14}\ \delta_{24} \dots \delta_{N4}]^\top$ to express the error dynamics in a compact form. Using (16) and (17), the time derivative of $\delta_{i1}(t)$, $\delta_{i2}(t)$, $\delta_{i3}(t)$ and $\delta_{i4}(t)$ can be found as

$$\begin{aligned} \dot{\delta}_{i1}(t) =& \delta_{i2}(t) - c_{ip}\left(1-a_{id}\right)\left(\delta_{i1}\left(t-\tau_i^{com}\right)\right. \\ & \left.-\delta_{(i+1)1}\left(t-\tau_i^{com}\right)\right) - \gamma_{ip} a_{id}\delta_{i1}(t), \end{aligned} \tag{19}$$

$$\begin{aligned} \dot{\delta}_{i2}(t) =& -c_{iv}\left(1-a_{id}\right)\left(\delta_{i2}\left(t-\tau_i^{com}\right)\right. \\ & \left.-\delta_{(i+1)2}\left(t-\tau_i^{com}\right)\right) - \gamma_{iv} a_{id}\delta_{i2}(t), \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{\delta}_{i3}(t) =& \delta_{i4}(t) - c_{ip}\left(1-a_{id}\right)\left(\delta_{i1}\left(t-\tau_i^{com}\right)\right. \\ & \left.-\delta_{(i+1)1}\left(t-\tau_i^{com}\right)\right) - \gamma_{ip} a_{id}\delta_{i1}(t), \end{aligned} \tag{21}$$

$$\dot{\delta}_{i4}(t) = -K_{ip}\delta_{i3}\left(t-\tau_i^{in}\right) - K_{iv}\delta_{i4}\left(t-\tau_i^{in}\right) + \dot{\delta}_{i2}(t). \tag{22}$$

Let us define $\delta(t) = \left[\delta_1^\top(t)\ \delta_2^\top(t)\ \delta_3^\top(t)\ \delta_4^\top(t)\right]^\top$ and hence errors (19), (20), (21), and (22) can be expressed in a more compact form as

$$\begin{aligned} \dot{\delta}(t) =& \begin{bmatrix} \Gamma_p & I_N & 0 & 0 \\ 0 & \Gamma_v & 0 & 0 \\ \Gamma_p & 0 & 0 & I_N \\ 0 & \Gamma_v & 0 & 0 \end{bmatrix} \delta(t) + \sum_{i=1}^{N}\left\{\begin{bmatrix} C_{ip} & 0 & 0 & 0 \\ 0 & C_{iv} & 0 & 0 \\ C_{ip} & 0 & 0 & 0 \\ 0 & C_{iv} & 0 & 0 \end{bmatrix}\right. \\ & \left.\times\,\delta\left(t-\tau_i^{com}\right)\right\} + \sum_{i=1}^{N}\left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{ip} & A_{iv} \end{bmatrix}\delta\left(t-\tau_i^{in}\right)\right\}. \end{aligned} \tag{23}$$

Here, all the matrices, $\Gamma_p, \Gamma_v, C_{ip}, C_{iv}, A_{ip}, A_{iv}$ are defined in (25). The characteristic quasipolynomial of (23) is conferred in (26).

$$\Gamma_p = \begin{bmatrix} -\gamma_{1p}a_{1d} & & & & \\ & \ddots & & & \\ & & -\gamma_{ip}a_{id} & & \\ & & & \ddots & \\ & & & & -\gamma_{Np}a_{Nd} \end{bmatrix}, \quad C_{ip} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & -c_{ip}\left(1-a_{id}\right) & c_{ip}\left(1-a_{id}\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad A_{ip} = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & -K_{ip} & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}, \tag{24}$$

$$\Gamma_v = \begin{bmatrix} -\gamma_{1v}a_{1d} & & & \\ & \ddots & & \\ & & -\gamma_{iv}a_{id} & \\ & & & \ddots \\ & & & & -\gamma_{Nv}a_{Nd} \end{bmatrix}, \quad C_{iv} = \begin{bmatrix} 0 & 0 & 0 & & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & -c_{iv}(1-a_{id}) & c_{iv}(1-a_{id}) & \cdots & & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & & \cdots & 0 & 0 \end{bmatrix}, \quad A_{iv} = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & -K_{iv} & \\ & & & \ddots \\ & & & & 0 \end{bmatrix}. \tag{25}$$

$$\Delta(s) = \det\left(\begin{bmatrix} sI_N - \Gamma_p - \sum_{i=1}^{N} C_{ip}e^{-s\tau_i^{com}} & -I_N & 0 & 0 \\ 0 & sI_N - \Gamma_v - \sum_{i=1}^{N} C_{iv}e^{-s\tau_i^{com}} & 0 & 0 \\ -\Gamma_p - \sum_{i=1}^{N} C_{ip}e^{-s\tau_i^{com}} & 0 & sI_N & -I_N \\ 0 & -\Gamma_v - \sum_{i=1}^{N} C_{iv}e^{-s\tau_i^{com}} & -\sum_{i=1}^{N} A_{ip}e^{-s\tau_i^{in}} & sI_N - \sum_{i=1}^{N} A_{iv}e^{-s\tau_i^{in}} \end{bmatrix}\right) \tag{26}$$

We simplify the characteristic quasipolynomial as

$$\begin{aligned} \Delta(s) &= \det\left(sI_N - \Gamma_p - \sum_{i=1}^{N} C_{ip}e^{-s\tau_i^{com}}\right) \\ &\times \det\left(sI_N - \Gamma_v - \sum_{i=1}^{N} C_{iv}e^{-s\tau_i^{com}}\right) \\ &\times \det\left(\begin{bmatrix} sI_N & -I_N \\ -\sum_{i=1}^{N} A_{ip}e^{-s\tau_i^{in}} & sI_N - \sum_{i=1}^{N} A_{iv}e^{-s\tau_i^{in}} \end{bmatrix}\right) \\ &= \bar{\rho}_a(s).\bar{\rho}_b(s).\bar{\rho}_c(s), \end{aligned} \tag{27}$$

with $\bar{\rho}_a(s) = \prod_{i=1}^{N}\left(s + \gamma_{ip}a_{id} + c_{ip}(1-a_{id})e^{-s\tau_i^{com}}\right)$, $\bar{\rho}_b(s) = \prod_{i=1}^{N}\left(s + \gamma_{iv}a_{id} + c_{iv}(1-a_{id})e^{-s\tau_i^{com}}\right)$, and $\bar{\rho}_c(s) = \prod_{i=1}^{N}\left(s^2 + K_{iv}se^{-s\tau_i^{in}} + K_{ip}e^{-s\tau_i^{in}}\right)$. Our control algorithm will converge if and only if all the roots of the quasipolynomials $\bar{\rho}_a(s)$, $\bar{\rho}_b(s)$, $\bar{\rho}_c(s)$ reside in the open left-half complex plane. According to Lemma 2, the conditions on controller gains has been found as $K_{ip}, K_{iv} > 0$, and

$$\tau_i^{in} < \frac{1}{\sqrt{\frac{K_{iv}^2 + \sqrt{K_{iv}^4 + 4K_{ip}^2}}{2}}} \tan^{-1}\sqrt{\frac{K_{iv}^2 + \sqrt{K_{iv}^4 + 4K_{ip}^2}}{2K_{ip}^2}} K_{iv}.$$

Therefore, with $K_{ip} = \frac{\beta_i}{2}K_{iv}^2$, $\rho_c(s)$ has roots in the open left-half complex plane if and only if

$$0 < K_{iv} < \frac{\tan^{-1}\left(\frac{2}{\beta_i}\sqrt{\frac{1+\sqrt{1+\beta_i^2}}{2}}\right)}{\tau_i^{in}\sqrt{\frac{1+\sqrt{1+\beta_i^2}}{2}}}, \ \beta_i > 0, \ \forall i. \tag{28}$$

The necessary and sufficient conditions for the roots of $\bar{\rho}_a(s)$, and $\bar{\rho}_b(s)$ to have negative real parts can be obtained following the proof of Theorem 1. We get the conditions on c_{ip} and c_{iv} as

$$\left.\begin{aligned} 0 < c_{ip}, c_{iv} < \frac{\pi}{2\tau_i^{com}}, &\quad \text{if} \quad a_{id} = 0, \\ \gamma_{ip}, \gamma_{iv} > 0, &\quad \text{if} \quad a_{id} = 1 \end{aligned}\right\}, \ \forall i. \tag{29}$$

Therefore, error $\delta(t)$ converge to zero if and only if (28) and (29) are satified. This means position and velocity of each agent converges to that of desired trajectory. ■

4. SIMULATION RESULTS

In this section, we show trajectory tracking for a group of 4 agents for both the single-integrator and double-integrator case. The desired trajectory in both the cases are same and given by $x_d(t) = 1 + 10\sin(t)$. Only agent 1 has the knowledge of the desired trajectory $x_d(t)$.

4.1 Illustration 1: Single-Integrator Case

The communication delays and input delays are heterogeneous. The initial position, input delays and communication delays for each agent are tabulated in Table 1. Without loss of generality, we can take initial estimates of the desired trajectory as zero.

Agents	State $x_i(t_0)$	Input delay τ_i^{in} (in seconds)	Communication delay τ_i^{com} (in seconds)
Agent 1	2	0.3	1
Agent 2	5	0.6	0.5
Agent 3	12	0.9	0.7
Agent 4	15	1.2	0.1

Table 1. Initial positions, input delays and communication delays in a group of 4 agents

According to Theorem 1, we select controller gains as $K_1 = 3.6652$, $K_2 = 1.8326$, $K_3 = 1.2217$, $K_4 = 0.9163$ and estimator gains as $c_2 = 2.1991$, $c_3 = 1.5708$, $c_4 = 10.9956$. As $x_d(t)$ is known to only agent 1, we have to ensure $\gamma_1 > 0$. We choose $\gamma_1 = 1$. The error between the position

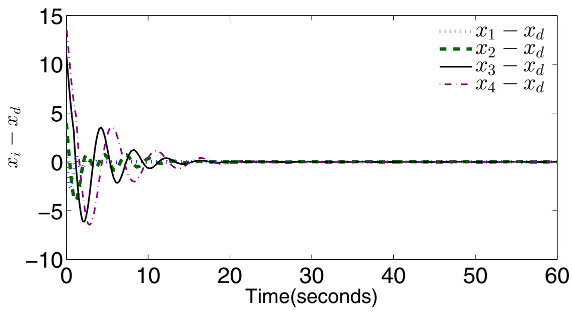

Fig. 1. Error between position of agents and $x_d(t)$.

of agents and the desired trajectory is shown in Fig. 1. It shows that the proposed control methodology ensures trajectory tracking.

4.2 Illustration 2: Double-Integrator Case

The initial position, input delays and communication delays are same as given in Illustration 4.1. The initial velocity for the 4-agent system is given by $v_1(t_0) = 3$, $v_2(t_0) = 5$, $v_3(t_0) = 6$, $v_4(t_0) = 9$. Initially, all the estimates are zero. According to Theorem 2, we design controller parameters as $K_{1v} = 2$, $K_{2v} = 1.5$, $K_{3v} = 1$, $K_{4v} = 0.5$, and $\beta_i = 1$, $\forall i$. The estimator gains are $c_{2p} = 1.5708$, $c_{3p} = 1.1220$, $c_{4p} = 7.8540$, $c_{2v} = 2.1991$, $c_{3v} = 1.5708$, $c_{4v} = 10.9956$. As only agent 1 has the knowledge of $x_d(t)$, we need to design γ_{1p} and γ_{1v}. We set these two parameters as 1. The error between the states of each agent and the desired trajectory are shown in Figs. 2 and 3. It can be that the errors decay to zero and all agents track the desired trajectory.

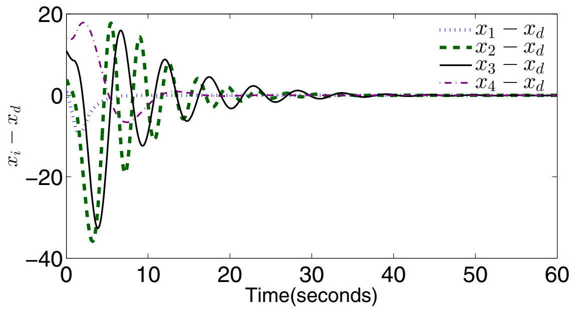

Fig. 2. *Error between position of agents and $x_d(t)$.*

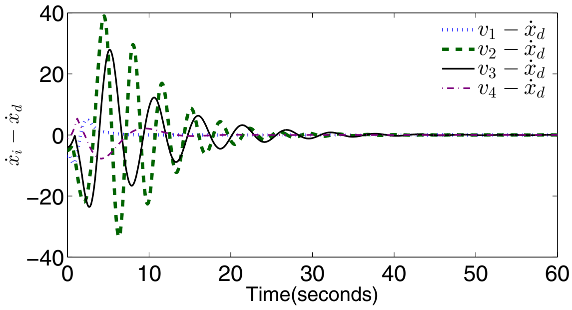

Fig. 3. *Error between velocity of agents and $\dot{x}_d(t)$.*

5. CONCLUSION

The effect of heterogeneous input delays and communication delays in the consensus tracking problem is investigated. Necessary and sufficient conditions are derived for the proposed protocol. Controller and estimator gains are designed independently and they have inverse relationship with input and communication delays respectively. In future, the effect of time-varying delays and information loss can be investigated

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