Further Remarks on Single-Delay and Multiple-Delay PR Protocols for Fast Consensus in a Large-Scale Network

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Abstract: The performance of Proportional-Retarded (PR) protocols in both single-delay and multiple-delay settings is studied in a large-scale consensus dynamics. The benefits of using these protocols are investigated specifically by analytically tuning them for fast consensus and investigating their control effort and noise mitigation characteristics. Benchmark analyses demonstrate that PR protocols can be more preferable over Proportional (P) and Proportional-Derivative (PD) protocols in network settings.

Keywords: Multi-agent systems, fast consensus, delay-based control, pole placement.

1. INTRODUCTION

With the emergence of low cost sensing, actuation, and computation platforms, we are already envisioning a future where multiple autonomous robots, or shortly “agents”, will collectively work together to perform certain missions. However, rendering the agents to autonomously work together still poses a number of challenges. Among others, three key issues broadly studied in the context of multi-agent system are related to the presence of noise in network settings (Hunt et al., 2012), time delays arising in exchange of information (Olfati-Saber and Murray, 2004), and the structure of the network, i.e., the graph underlying the interactions between agents in the network (Schöllig et al., 2007; Qiao and Sipahi, 2016). In most studies these issues were addressed separately but not in combination although all three are interrelated with one another.

While many studies focus on the stability of multi-agent systems, authors also recognized the need of achieving certain performance characteristics (Carli et al., 2011). In the presence of delays, achieving fast stabilization in multi-agent systems is not straightforward mainly because highly-aggressive control actions can destabilize the system. Moreover, designing such systems for fast stabilization is challenging as there are too many parameters to tune, and infinite dimensionality of the corresponding eigenvalue problem due to delays makes this design even more challenging (Qiao et al., 2013). From a control point-of-view, rigorous control design tools to achieve fast consensus, especially for large-scale systems, still do not exist in the literature. One opportunity, as demonstrated for low order systems, is to utilize reliable computational tools to approximate system's rightmost roots (Vyhlídal and Zítek, 2009), or to use such tools to tune the controller gains via optimization (Michiels and Vyhlídal, 2005). Another opportunity is to take advantage of implementations of proportional-retarded (PR) controllers demonstrated for single-input single-output (SISO) systems (Ramírez et al., 2013; Suh and Bien, 1979; Abdallah et al., 1993; Selivanov and Fridman, 2017), and expand PR controllers for multi-agent systems. Some recent results along these lines include (Atay, 2013; Cao and Ren, 2010; Li et al., 2010; Yu et al., 2013; Meng et al., 2013; Song et al., 2016; Huang et al., 2016; Ramírez and Sipahi, 2018a; Fridman and Shaikhet, 2017).

The use of PR controllers for large-scale networked systems can be promising, although it still remains under-explored. Our recent results provide some guidelines as to how to address this on a widely-studied consensus dynamics (Ramírez and Sipahi, 2018b). Here, we first summarize from the cited study, mainly by demonstrating the analytical tuning of PR controllers with single and multiple delays. This tuning, especially in the multiple delay case, enables placing the system rightmost poles at a user defined spectral abscissa with a user-defined spectrum separation from the rest of the poles, for effective pole placement. One open question, which is the focus of this manuscript, is regarding how the PR controllers perform in terms of control effort and under noisy measurements, and how they compare with standard Proportional or Proportional-Derivative (P or PD) controllers. Here, we present a systematic approach to setup benchmark studies to establish this comparison, mainly in terms of metrics associated with agents' settling times, control effort, and noise sensitivity. Results suggest that multiple-delay PR controllers for the consensus dynamics at hand can render superior characteristics in terms of noise attenuation, while requiring slightly less control effort compared to the P and PD cases. This therefore suggests that PR controllers can be designed and utilized as better alternatives over standard P and PD-type controllers in network settings.

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2. PRELIMINARIES

We start with a system with n identical agents

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

where x_i is the state of agent i and u_i its control input.The topology of the network in (1) is described by an undirected graph $\mathcal{G} = (N, E)$ where $N = \{1, 2, \dots, n\} \equiv \overline{1, n}$ is the set of nodes and $E \subset N \times N$ is the set of edges. Each edge has a weight $a_{ij} = a_{ji}$, where the edge $(i, j) \in E$ indicates that agent i receives information from agent j if $a_{ij} \neq 0$. The Laplacian matrix $\mathbf{L} = [-a_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{G}$ accepts the property $\sum_{j=1}^{n} a_{ij} = 0$ for all $i \in N$. Hence, according to the spectral theorem for Hermitian matrices (Horn and Johnson, 1988), all the eigenvalues of $\mathbf{L}$ are real.

Remark 1. Assuming that the agents are connected, $\mathbf{L}$ has a zero eigenvalue $\lambda_1 = 0$ and the remaining eigenvalues for $a_{ij} > 0$, $i \neq j$ are positive (Olfati-Saber and Murray, 2004). Hereafter, we present the developments only on the case of unique eigenvalues as the extension to the case of repeated eigenvalues is straightforward. We adopt the convention $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n$.

Given m graphs sharing the same set of nodes $\mathcal{G}_m = (N, E_m)$ with associated Laplacian $\mathbf{L}_m = [-a_{m,ij}] \in \mathbb{R}^{n\times n}$, we define the composed Laplacian $\mathbf{L} := \sum_{m\in\mathcal{M}} \mathbf{L}_m$ where the m-th graph layer belongs to a finite index set $\mathcal{M} = 1, \dots, m \leq n$. Moreover, the neighbors of agent i, in the m-th layer, are denoted by $\mathcal{N}_i^m = \{j \in N : (i, j) \in E_m\}$. The control objective is to achieve agreement of the states amongst all the agents. To this end, here we introduce a distributed PR protocol for all the agents and subject to network topology,

$$u_i(t) = \sum_{m\in\mathcal{M}} \sum_{j\in\mathcal{N}_i^m} a_{m,ij} \left[k_{p_m} \Delta x_{ji}(t) - k_{r_m} \Delta x_{ji}(t - h_m)\right], \tag{2}$$

where $\Delta x_{ji}(t) = x_j(t) - x_i(t)$, the heterogenous coupling strengths $a_{m,ij}$ satisfy $a_{ij} = \sum_{m=1}^{n} a_{m,ij}$, and a_{ij} are the entries of $\mathbf{L}$. The heterogenous gains $k_{p_m} > 0$ and $k_{r_m} > 0$ determine respectively the strength of the proportional and retarded actions, and $h_m \geq 0$ are intentional multiple delays induced in the input to an agent.

As shown successfully in the literature, intentional delays can create realizable derivative effects to enhance performance (Suh and Bien, 1979). Here, we aim to investigate this opportunity for the large-scale consensus dynamics (1)-(2). We state that protocol (2) solves the consensus problem if $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ for all $i, j \in N$.

Let $\mathbf{x} = \begin{pmatrix} x_1 \cdots x_n \end{pmatrix}^\top$ be the stack vector of the states at all nodes, then system (1)-(2) is written in matrix form as

$$\dot{\mathbf{x}}(t) = \sum_{m\in\mathcal{M}} \mathbf{L}_m[-k_{p_m}\mathbf{x}(t) + k_{r_m}\mathbf{x}(t - h_m)]. \tag{3}$$

In the following we present two case studies. The first case considers a complete graph with a single layer; i.e., $\mathcal{M} = 1$. The second case, on the other hand, investigates the use of n graph-layers; that is, $\mathcal{M} = \overline{1, n}$.

3. ANALYTICAL TUNING OF THE PR PROTOCOL

Delay-based control is an effective alternative to benchmark control schemes (Ramírez et al., 2016). Based on (2), we next present the single and multiple delay PR protocols and summarize from (Ramírez and Sipahi, 2018b) how to analytically assign system's spectral abscissa at a desired locus γ_d. This approach is scalable and easy to implement, and can achieve a desired separation, DI_d, between system rightmost poles and rest of the spectrum, consistent with what has been advocated in the literature as a critical design constraint (Ramírez et al., 2017; Zítek et al., 2013), see also (Ramírez and Sipahi, 2018a,b) for details.

Case I. Single-delay PR protocol: The single-delay version of the PR protocol considers a complete graph with a single layer, hence $\mathcal{M} = 1$ and $\mathcal{N}_i^1 = 1, \dots, n$. Then, the single-delay PR protocol is readily obtained from (2) as

$$u_i(t) = \sum_{j=1}^{n} a_{1,ij}[k_{p_1}\Delta x_{ji}(t) - k_{r_1}\Delta x_{ji}(t - h_1)]. \tag{4}$$

Matrix form of (1) and (4) follows from (3) as

$$\dot{\mathbf{x}}(t) = \mathbf{L}_1[-k_{p_1}\mathbf{x}(t) + k_{r_1}\mathbf{x}(t - h_1)]. \tag{5}$$

In this setting $\mathbf{L}_1$ is symmetric, hence the Schur's theorem (Horn and Johnson, 1988) guarantees the existence of a nonsingular orthogonal matrix $\mathbf{U} \in \mathbb{R}^{n\times n}$, such that $\mathbf{L}_1 = \mathbf{U}\mathbf{D}_1\mathbf{U}^{-1}$ holds, where $\mathbf{D}_1 = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$. Then, the change of variable $\mathbf{x}(t) = \mathbf{U}\boldsymbol{\xi}(t)$ transforms system (5) into

$$\dot{\boldsymbol{\xi}}(t) = \mathbf{D}_1[-k_{p_1}\boldsymbol{\xi}(t) + k_{r_1}\boldsymbol{\xi}(t - h_1)]. \tag{6}$$

This diagonal form allows obtaining the following tuning:

Proposition 2. (Ramírez and Sipahi (2018b)). Given a desired spectral abscissa $\gamma_d < 0$ and a desired dominance index $\mathrm{DI}_d \in (1, 2.3102)$. Then, for network (5), a dominant root at γ_d is placed by distributing the spectrum of $\mathbf{L}_1$ as

$$\lambda_m = -\frac{\gamma_d \Delta_m(\mathrm{DI}_d - 1)}{\Omega_0(\mathrm{DI}_d\Omega_0 - \mathrm{DI}_d + 1)}, \quad m = \overline{1, n}, \tag{7}$$

and tuning the gains of the PR protocol as

$$(h_1, k_{p_1}, k_{r_1}) = \left(\frac{-\Omega_0}{\gamma_d(\mathrm{DI}_d - 1)}, \frac{\Omega_0 \mathrm{DI}_d}{\mathrm{DI}_d - 1}, e^{-k_{p_1}}\right), \tag{8}$$

where $\Delta_m = \mathrm{DI}_d\Omega_0(\delta_{m-1} + m - 2) + \delta_{m-2}(\mathrm{DI}_d\Omega_0 - \mathrm{DI}_d + 1)$, the constant [1] $\Omega_0 = 0.5671$, and δ_{m-m_0} is the Kronecker delta function [2]. Moreover, this dominant root is isolated from the rest of the spectrum by $\mathrm{DI}_d \times \gamma_d$. □

Case II. Multiple-delay PR protocol: The multiple-delay version of the PR protocol uses n graph-layers; i.e., $\mathcal{M} = \overline{1, n}$. Here, the composed Laplacian $\mathbf{L} = \sum_{m=1}^{n} \mathbf{L}_m$ is restricted to a symmetric form, hence $a_{m,ij} = a_{m,ji}$. Moreover the network must be strongly connected. Then, the multiple-delay PR protocol follows from (2) as

$$u_i(t) = \sum_{m=1}^{n}\sum_{j=1}^{n} a_{m,ij}[k_{p_m}\Delta x_{ji}(t) - k_{r_m}\Delta x_{ji}(t - h_m)]. \tag{9}$$

Matrix form of (1) and (9) is in this case obtained as

$$\dot{\mathbf{x}}(t) = \sum_{m=1}^{n} \mathbf{L}_m[-k_{p_m}\mathbf{x}(t) + k_{r_m}\mathbf{x}(t - h_m)], \tag{10}$$

whose diagonal representation, under the change of variable $\mathbf{x}(t) = \mathbf{U}\boldsymbol{\xi}(t)$, is given by

$$\dot{\boldsymbol{\xi}}(t) = \sum_{m=1}^{n} \mathbf{D}_m[-k_{p_m}\boldsymbol{\xi}(t) + k_{r_m}\boldsymbol{\xi}(t - h_m)]. \tag{11}$$

Here $\mathbf{U} \in \mathbb{R}^{n\times n}$ is a nonsingular orthogonal matrix, such that $\mathbf{L} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1} = \sum_{m=1}^{n} \mathbf{U}\mathbf{D}_m\mathbf{U}^{-1} = \sum_{m=1}^{n} \mathbf{L}_m$ holds, and $\mathbf{D} = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$, $\mathbf{D}_m = \lambda_m\mathbf{J}_{mm}$ and $\mathbf{J}_{mm} \in \mathbb{R}^{n\times n}$ is a single-entry matrix whose (m, m)th entry is one. Based on the above decomposition, the parameters of the PR controller can be tuned:

[1] The constant Ω_0 follows from the principal branch of the Lambert W function as $\Omega_0 \equiv W_0(1)$, see also (Ramírez and Sipahi, 2018b).

[2] Function $\delta_{m-m_0} = \delta[m - m_0]$ follows the standard definition; i.e., $\delta[m - m_0] \equiv 1$ if $m = m_0$ and $\delta[m - m_0] \equiv 0$ if $m \neq m_0$.

Proposition 3. (Ramírez and Sipahi (2018b)). Given a desired exponential decay rate $\gamma_d < 0$ and a desired dominance index $\mathrm{DI}_d > 1$ for the network (10), then a dominant root at γ_d is placed by distributing the spectrum of $\mathbf{L}$ as

$$\lambda_m = \gamma_d(m-1)(\mathrm{DI}_d - 1)/\Omega, \quad m = \overline{1,n}, \tag{12}$$

and tuning the gains of the multiple-delay PR protocol as

$$h_m = \frac{\Omega}{\gamma_d(m-1)(\mathrm{DI}_d-1)}, \tag{13}$$

$$k_{p_m} = \Omega_0 - \Omega\frac{\Delta_m(\mathrm{DI}_d-1)+1}{(m-1)(\mathrm{DI}_d-1)}, \quad m = \overline{2,n}, \tag{14}$$

$$k_{r_m} = e^{-k_{p_m}}, \tag{15}$$

with $(h_1, k_{p_1}, k_{r_1}) = (0,0,0)$. Here, $\Delta_m = (m-2)(m-1)/2$, with[3] $\Omega = -2.1011$, and $\Omega_0 = 0.5671$. This dominant root is isolated from the rest of the spectrum by $\mathrm{DI}_d \times \gamma_d$. □

The main difference between Propositions 2 and 3 is that the multiple-delay PR protocol enables an arbitrary separation DI_d between the rightmost roots and the rest of the spectrum. On the other hand, the single-delay PR protocol not only requires fewer parameters to design but it can also be implemented on graphs that are not necessarily strongly connected.

Graph generation: Matrices $\mathbf{D}_1 = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ and $\mathbf{D}_m = \lambda_m \mathbf{J}_{mm}$ are completely determined from Propositions 2 and 3. Hence, with $\mathbf{L}_1 \to \mathbf{U}^{-1}\mathbf{L}_1\mathbf{U}$ and $\mathbf{L}_m \to \mathbf{U}^{-1}\mathbf{L}_m\mathbf{U}$, one obtains respectively $\mathbf{L}_1$ and $\mathbf{L}_m$. The unitary matrix $\mathbf{U}$ is found by constructing an orthonormal basis $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, where $\mathbf{u}_m \in \mathbb{R}^n$ is the eigenvector associated with λ_m. In addition, the graph Laplacian has always a right eigenvector $\mathbf{1}_n = (1, \ldots, 1)^\top$ corresponding to its zero eigenvalue. Without loss of generality, propose $\tilde{\mathcal{B}} = \{\mathbf{1}_n, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_m$ has a single non-zero entry with value 1 in the mth position. The orthonormal basis is then retrieved using the Gram-Schmidt process (Horn and Johnson, 1988; Leon et al., 2013; Qiao et al., 2013).

4. NUMERICAL EVALUATION

We present a comparison case where the single-delay PR protocol outperforms a P and PD protocols respectively in terms of converge rate and noise attenuation. Then, we illustrate the advantages of using multi-delays over the use of single-delay[4]. To estimate the convergence rate we measure the settling time t_s, defined by an $\varepsilon\%$ settling rule on the total displacement of all the agents, $|\mathbf{x}| = \left[\sum_{i=1}^n x_i^2(t)\right]^{1/2}$. The $\varepsilon\%$ settling criterion is based on the total drop of $|\mathbf{x}|$ between its initial and final values. Here, ε may be interpreted as a settling error. Then, $\varepsilon \to 0$ indicates that t_s is measured when $|\mathbf{x}| \to |\mathbf{x}_\alpha|$, where $|\mathbf{x}_\alpha|^2 = n\alpha^2$, and $\alpha = 1/n\sum_{i=1}^n x_i(0)$ is the average-consensus defined by the agents' initial states $x_i(0)$. Further, we need to contrast speed of convergence against effort. To this end, we estimate the total control effort using $\mathrm{TCE} = \int_0^{t_s} |\mathbf{u}|dt$, where $|\mathbf{u}| = \left[\sum_{i=1}^n u_i^2(t)\right]^{1/2}$.

[3] The constant Ω follows from the principal and the first branch of the Lambert W function as $\Omega \equiv \Re(W_1(1) - W_0(1))$, see also (Ramírez and Sipahi, 2018b).

[4] The results are based on time simulations in Simulink with ODE1 solver and fixed-step size of 1 ms.

Table 1. S1, delays and Laplacian eigenvalues, $\mathrm{DI}_d = 1.5$

$-\gamma_d$	h_1	λ_1	λ_2	λ_3	λ_4	λ_5
1.5	0.7562	0	1.3224	3.2077	6.4155	9.6232
2	0.5671	∗	1.7632	4.2770	8.5540	12.8309
2.5	0.4537	∗	2.2040	5.3462	10.6924	16.0387
3	0.3781	∗	2.6448	6.4155	12.8309	19.2464

Table 2. S1, TCE and t_s based on a 2% rule

PR protocol, Eq. (4)			P protocol, Eq. (16)		
$-\gamma_d$	t_s	TCE	$-\gamma_d$	t_s	TCE
1.5	1.59	1.1329	1.5	1.885	1.0836
2	1.192	1.1273	2	1.413	1.08
2.5	0.953	1.1226	2.5	1.130	1.0763
3	0.794	1.1163	3	0.941	1.0727

4.1 Single-delay PR protocol versus P(PD) protocol

To form a common basis for comparison, we use the following P protocol as a benchmark,

$$u_i(t) = \textstyle\sum_{j=1}^n a_{1,ij}[r_{p_1}\Delta x_{ji}(t)], \tag{16}$$

which is obtained by removing the retarded term in (4). Due to the fact that $\mathbf{L}_1$ is diagonalizable, the characteristic equation of (1) with (16) is given by the product of the factors $f_m(s) = s + \lambda_m r_{p_1}$, $m = \overline{1,n}$. Since the Laplacian eigenvalues are assumed to satisfy $\lambda_1 < \cdots < \lambda_n$, it can be proved that the spectral abscissa of the consensus network (1) with (16) can be assigned at γ_d using

$$r_{p_1} = -\gamma_d/\lambda_2 = \Omega_0/(\mathrm{DI}_d - 1), \tag{17}$$

thus providing a comparable dynamic response with respect to the PR protocol. It is worthy to mention that the case associated with $\lambda_1 = 0$ can be neglected as this pole corresponds only to the consensus state of the dynamics.

Simulation 1 (S1): For the five-agent problem at hand, the Gram-Schmidt process yields the orthonormal basis $\mathcal{B} = \{\mathbf{1}_5\sqrt{5}/5, (5\mathbf{e}_2 - \mathbf{1}_5)\sqrt{5}/10, (4\mathbf{e}_3 + \mathbf{e}_2 - \mathbf{1}_5)\sqrt{3}/6, (2\mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_1)\sqrt{6}/6, (\mathbf{e}_5 - \mathbf{e}_1)\sqrt{2}/2\}$ with which the unitary matrix $\mathbf{U}$ is constructed. Given γ_d and DI_d, the Laplacian eigenvalues can be computed from (7) and then employed to form $\mathbf{D}_1 = \mathrm{diag}\{\lambda_1, \ldots, \lambda_5\}$. With $\mathbf{U}$ and $\mathbf{D}_1$ at hand, the corresponding Laplacian matrix follows from $\mathbf{L}_1 = \mathbf{U}\mathbf{D}_1\mathbf{U}^{-1}$. With $\mathbf{L}_1$ computed, it is easy to see that we obtain a fully connected graph[5]. Finally, the gains of PR and P protocols follow respectively from (8) and (17).

In $S1$, we test the convergence rate of the above defined five-agent network subject to both PR and P protocols for an initial condition satisfying $\alpha = 0$ with $\mathrm{DI}_d = 1.5$ and $\gamma_d \in \{-1.5, -2, -2.5, -3\}$. The computed Laplacian eigenvalues are listed in Table 1. Since DI_d is kept fixed, the gains $k_{p_1} = 1.7014$, $k_{r_1} = 0.1824$ and $r_{p_1} = 1.1343$ remain the same for all the considered γ_d. On the other hand, delay h_1 decreases with smaller γ_d values as shown in the same table. Fig. 1 shows the total displacement of the agents with a 2% settling time rule and Table 2 summarizes the performance indices. Indeed, and as expected, smaller values of γ_d are associated with faster responses. Clearly, the single-delay PR solves the consensus problem faster than P but requires slightly more control effort.

[5] The design can be adapted to a given sparse graph (Ramírez and Sipahi, 2018b) as PR gains scale only a single Laplacian, see (3).

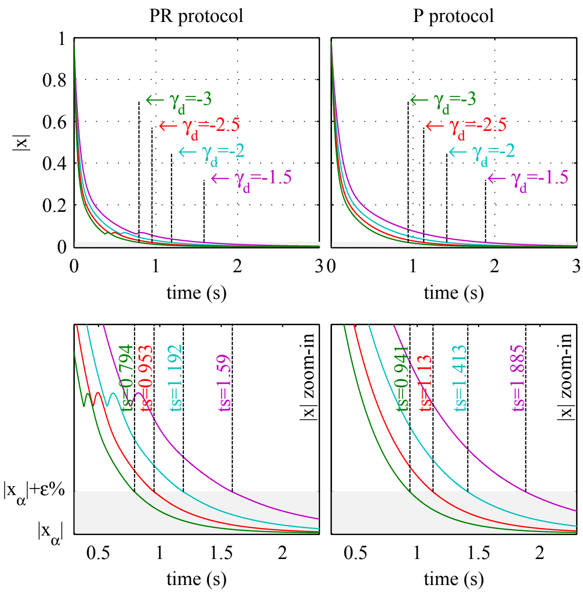

Fig. 1. S1; (Left panels) Single-delay PR protocol (4). (Right panels) P protocol (16). (Top panels) Total displacement and settling time, initial condition $(0.6, 0.3, -0.2, -0.7, 0)^\top$. (Bottom panels) $|\mathbf{x}|$ zoom-in and 2% settling time envelope in gray.

Simulation 2 (S2): We now examine how consensus velocity and control effort vary with the size of the network n and its initial conditions. For different n ranging from 10 to 100 agents with increments of 10, we perform 1000 trials for each n. For each trial the initial conditions and the desired dominance index are re-randomized and seeded by computer clock while $\gamma_d = -1.5$ is kept fixed. The result in Fig. 2 summarizes the ratio of the means of t_s and TCE obtained in the multiple-delay PR case to the P case. These ratios being less than unity indicate that the PR network outperforms the P network. Clearly, the PR network exceeds the convergence speed of the P network however requiring larger, on average, control effort.

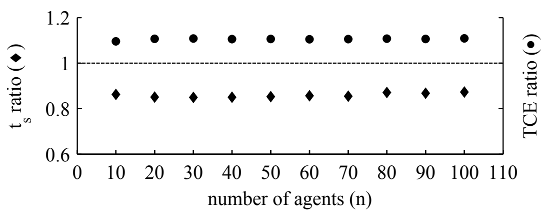

Fig. 2. S2; Mean t_s and TCE ratios for $\gamma_d = -1.5$ where for each n, 1000 trials are performed. Initial conditions and DI_d are re-randomized in each trial and drawn from a uniform distribution in $(-1, 1)$ and $(1.5, 1.8)$, respectively.

Simulation 3 (S3): As a well-known rule of thumb, improved transient dynamics in systems, in terms of convergence rates, can be achieved with the predictive nature of the feedback controller. In classical control, this corresponds to the use of derivative control actions, which are known to provide sufficient damping and high reactivity on the system. With this in mind, aiming at speeding up convergence speed of the P network, we next complement (16) with a Derivative (D) control action and obtain the PD protocol

$$u_i(t) = \textstyle\sum_{j=1}^{n} a_{1,ij}[r_{p_1}\Delta x_{ji}(t) + r_{d_1}\Delta\dot{x}_{ji}(t)], \qquad (18)$$

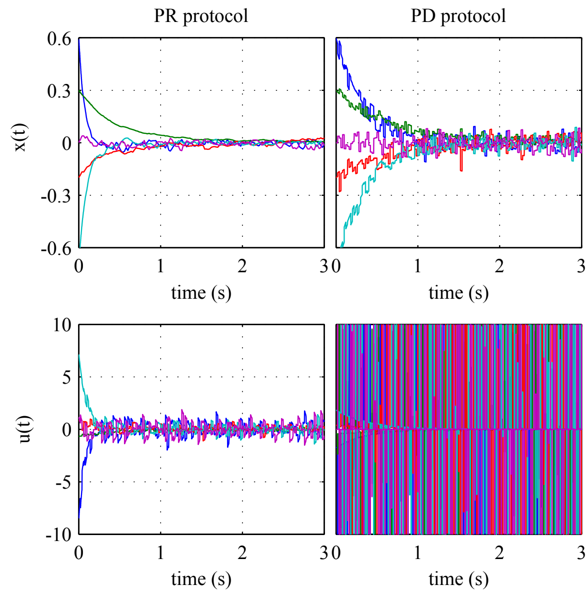

Fig. 3. S3; (Left panels) Single-delay PR protocol (4). (Right panels) PD protocol (18). (Top panels) Agents' states, initial condition $(0.6, 0.3, -0.2, -0.7, 0)^\top$. (Bottom panels) Control signals.

where r_{p_1} and r_{d_1} are the proportional and derivative gains, respectively. Following the same decomposition as in the case of the PR controller, the characteristic equation of system (1) with (18) factorizes as $f_m(s) = s + \lambda_m r_{p_1}/(1 + \lambda_m r_{d_1})$. Then, using the gains

$$(r_{p_1}, r_{d_1}) = \left(\frac{\gamma_d \mathrm{DI}_d(\lambda_2 - \lambda_3)}{\lambda_2\lambda_3(\mathrm{DI}_d - 1)}, \frac{\lambda_3 - \mathrm{DI}_d\lambda_2}{\lambda_2\lambda_3(\mathrm{DI}_d - 1)}\right), \qquad (19)$$

the first two rightmost roots of the PD network are placed at the same locus with those of the PR network provided that $\mathrm{DI}_d < \lambda_3(\lambda_2 - \lambda_n)\lambda_2^{-1}(\lambda_3 - \lambda_n)^{-1}$, thus expected to yield a comparable response.

In $S3$, we use the 5-agent network defined in $S1$ with $\gamma_d = -1.5$ and $\mathrm{DI}_d = 1.5$, where we have injected uniformly distributed random signals in the communication channels to mimic high-frequency noise measurements of the states with a flat power spectral density and infinite total energy. The result is displayed in Fig. 3. Two observations are in order: i) agents' dynamics of the PR network are minimally affected by the simulated high-frequency noise in the measurements as opposed to using the PD protocol and ii) the network with the PR controller yields a much smoother control signal compared to the PD controller thus reducing actuator chattering. Consequently, the control designer is advised to utilize PR over PD protocols when dealing with noisy measurements.

4.2 Multiple-delay PR protocol versus P protocol

Similarly, we remove the retarded terms in (9) to obtain the P protocol with heterogenous gains

$$u_i(t) = \textstyle\sum_{m=1}^{n}\sum_{j=1}^{n} r_{p_m}[a_{m,ij}\Delta x_{ji}(t)], \qquad (20)$$

with which the dynamic properties of PR in (9) are to be compared. In this case, the characteristic equation of (1) with (20) is given by the factors $f_m(s) = s + \lambda_m r_{p_m}$. It is worthy of mention that the spectral abscissas of the PR subsystems are placed at $\gamma_{d_m} = \Delta_m(\mathrm{DI}_d - 1)\gamma_d + \gamma_d$. It follows from γ_{d_m} and λ_m in (12) that choosing

$$r_{p_m} = -\frac{\gamma_{d_m}}{\lambda_m} = -\Omega\frac{\Delta_m(\mathrm{DI}_d - 1) + 1}{(m-1)(\mathrm{DI}_d - 1)}, \quad m = \overline{2, n}, \quad (21)$$

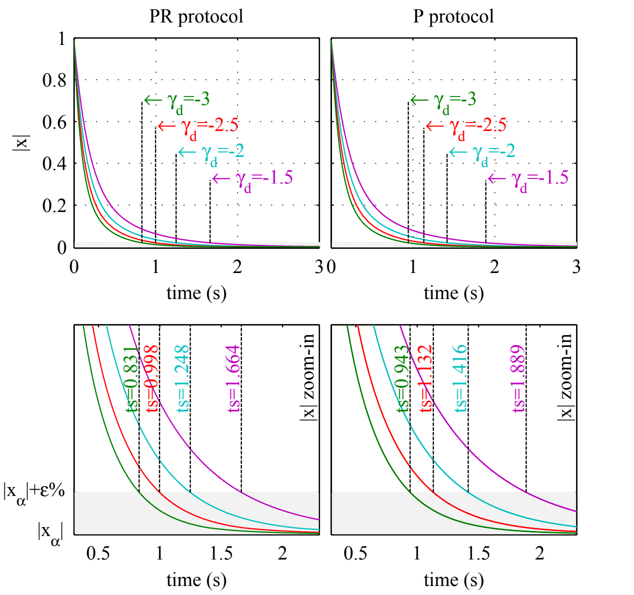

Fig. 4. S4; (Left panels) Multiple-delay PR protocol (9). (Right panels) P protocol (20). (Top panels) Total displacement and settling time, initial condition $(0.6, 0.3, -0.2, -0.7, 0)^\top$. (Bottom panels) $|\mathbf{x}|$ zoom-in and 2% settling time envelope in gray.

Table 3. S4, Laplacian eigenvalues, $\mathrm{DI}_d = 1.5$

$-\gamma_d$	λ_1	λ_2	λ_3	λ_4	λ_5
1.5	0	0.3570	0.7139	1.0709	1.4279
2	∗	0.4760	0.9519	1.4279	1.9038
2.5	∗	0.5949	1.1899	1.7848	2.3798
3	∗	0.7139	1.4279	2.1718	2.8557

and $r_{p_1} = 0$, the spectral abscissas of the P subsystems are placed at the same locus with those of the PR subsystems, thus providing a comparable response. Once again, $\lambda_1 = 0$ can be neglected as noted above.

Simulation 4 (S4): We now use the multiple-delay PR protocol in $S1$. With $\gamma_d \in \{-1.5, -2, -2.5, -3\}$ and $\mathrm{DI}_d = 1.5$ in (12), we compute the Laplacian eigenvalues in Table 3. Since DI_d is kept fixed, the gains k_{p_m}, k_{r_m}, and r_{p_m} remain the same for all considered γ_d while the multiple delays h_m decrease with smaller γ_d values, see Tables 4 and 5. With the Laplacian eigenvalues at hand, we next construct the matrices $\mathbf{D}_m = \lambda_m \mathbf{J}_{mm}$ as in Proposition 3 and use them along with $\mathbf{U}$ to reveal the Laplacian forms $\mathbf{L}_m = \mathbf{U}\mathbf{D}_m\mathbf{U}^{-1}$, and finally obtain the composed Laplacian matrix $\mathbf{L} = \sum_{m=1}^{n} \mathbf{L}_m$. After computations, we observe that the design technique lands itself into a fully connected graph.

Fig. 4 shows the total displacement of the agents with a 2% settling rule. Here, average consensus is reached asymptotically where in this case the group disagreement decreases monotonically for both protocols as $|\mathbf{x}|$ vanishes. Table 6 summarizes the performance indices from which we can see that PR outperforms P in terms of convergence ratios and moreover PR requires slightly less control effort than that required in the P network.

Simulation 5 (S5): Now we demonstrate the speed of convergence of the algorithms (9) and (20) for two different networks with $n = 10$ and $n = 100$, and randomized initial states satisfying $\alpha = 0$. The state trajectories of the systems

Table 4. S4, delay values, $\mathrm{DI}_d = 1.5$

$-\gamma_d$	h_1	h_2	h_3	h_4	h_5
1.5	0	2.8014	1.4007	0.9338	0.7004
2	∗	2.1011	1.0505	0.7004	0.5253
2.5	∗	1.6806	0.8404	0.5603	0.4202
3	∗	1.4007	0.7004	0.4669	0.3502

Table 5. S4, gain values, $\mathrm{DI}_d = 1.5$

m	1	2	3	4	5
k_{p_m}	0	4.7693	3.7187	4.0689	4.7693
k_{r_m}	∗	0.0085	0.0243	0.0171	0.0085
r_{p_m}	∗	4.2021	3.1516	3.5018	4.2021

Table 6. S4, TCE and t_s based on a 2% rule

PR protocol, Eq. (9)			P protocol, Eq. (20)		
$-\gamma_d$	t_s	TCE	$-\gamma_d$	t_s	TCE
1.5	1.664	1.0298	1.5	1.889	1.0302
2	1.248	1.0275	2	1.416	1.0282
2.5	0.998	1.0253	2.5	1.132	1.0263
3	0.831	1.0230	3	0.943	1.0243

and the corresponding settling times are shown in the top panels of Figs. 5 and 6 for $n = 10$ and $n = 100$, respectively. Observe that consensus is reached about 12% faster when multiple-delays are used. Further, in contrast with the single-delay case, comparable TCE is required by both PR and P protocols in the selected networks.

Simulation 6 (S6): Finally, we examine how consensus velocity and control effort vary with the size of the network and its initial conditions. Here, for different number of nodes $n \in \{5, 10, 25, 50\}$, 1000 trials are executed for each n, where for each trial $\gamma_d \in (-2, -1.5)$, $\mathrm{DI}_d \in (1.5, 2)$, and $x_i(0) \in (-1, 1)$ are re-randomized and seeded by computer clock. The result is summarized in Fig. 7, which shows the ratio of the means of t_s and TCE obtained in the multiple-delay PR case to the P case. In the multiple-delay case, we see that the PR network consistently exceeds convergence speed of the P network with comparable control effort.

5. CONCLUSIONS

Performance of single-/multiple-delay PR protocols in comparison to benchmark P and PD implementations are assessed in a multi-agent system by way of simulations considering settling time of the agent dynamics, control effort, and noise rejection capabilities. We report that PR protocols not only require comparable control effort while achieving satisfactory speed of reach to consensus, but they can also successfully handle noisy measurements without the need of additional filtering. These results indicate that these protocols can be better alternatives over benchmarks in network settings.

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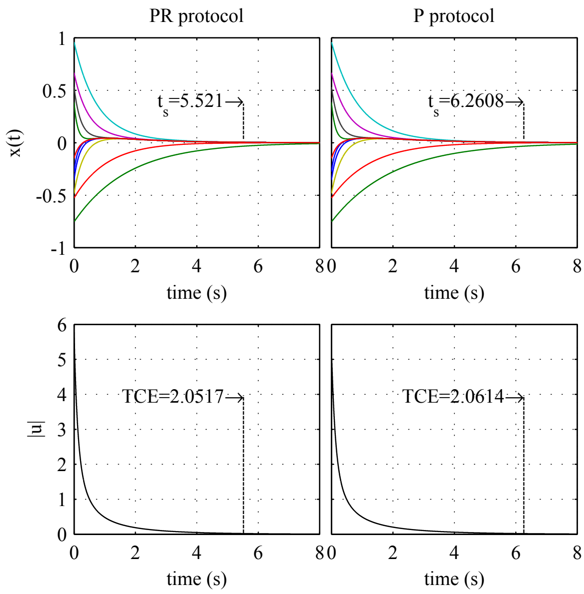

Fig. 5. S5; 10-agent network: Agents' states and control effort, $\gamma_d = -0.5$ and $\mathrm{DI}_d = 1.5$. Random initial conditions with zero-mean drawn from a uniform distribution in $(-1, 1)$ and t_s measured with a 2% rule. (Left panels) Multiple-delay PR protocol (9). (Right panels) P protocol (20).

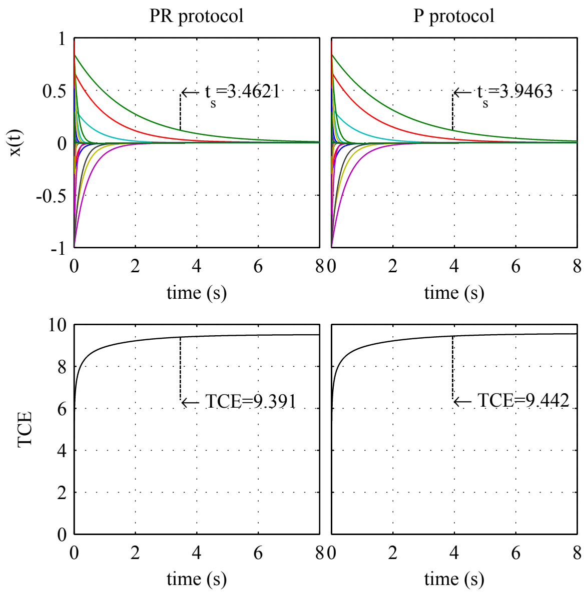

Fig. 6. S5; 100-agent network: Agents' states and total control effort, $\gamma_d = -0.5$ and $\mathrm{DI}_d = 1.5$. Random initial conditions with zero-mean drawn from a uniform distribution in $(-1, 1)$ and t_s measured with a 2% rule. (Left panels) Multiple-delay PR protocol (9). (Right panels) P protocol (20).

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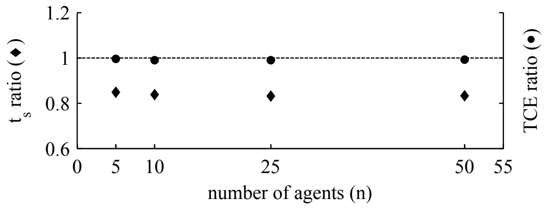

Fig. 7. S6; mean t_s and TCE ratios where for each n, 1000 trials are performed. Initial conditions, γ_d and DI_d are re-randomized in each trial and drawn from a uniform distribution in $(-1, 1)$, $(-2, -1.5)$ and $(1.5, 2)$, respectively.

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