Approximation-based Transient Behavior Analysis of Multi-Agent Systems with Delay

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Abstract: This paper proposes an approximation-based analysis method for the asymptotic behavior of a class of multi-agent systems with communication delay among the agents. It is considered that the agents implement the consensus protocol. The communication delay affects the asymptotic behavior of the agents that cannot be analyzed by using the methods developed for ordinary differential equations. The method proposed in this paper explores that, if the delay satisfies a "smallness" condition, then the delayed system has a number of dominant eigenvalues, which can be computed using numerical methods. The result of the proposed approximation method is a delay-free multi-agent system with weighted consensus protocol which has the same order as the original system, and it approximates well the dynamic behavior of the original multi-agent system with communication delay.

Keywords: Time delay, Differential equations, Asymptotic properties, Multi-agent systems

1. INTRODUCTION

Multi-Agent System (MAS) based coordination methods have made significant progress in the last years due to the development of the communication technology, robotics, and computer science (Mesbahi and Egerstedt, 2010). Many multidisciplinary research branches, such as control theory, biology, and statistical physics, gave major attention to these methods due to their broad applicability in many fields (Vicsek et al., 1995). These include sensor networks, reference tracking robot groups, etc.

The MAS uses some form of communication among the agents. With the increase in the number of agents and the physical distance between the agents, the communication delay becomes significant, and it cannot be neglected. As such, the behavior of MAS must be analyzed by taking into consideration the time delay induced by the communication.

Although, the stability of MASs can be assured in the presence of the communication lag (Liu and Liu, 2017), the delay influences the dynamic behavior of these systems (Michiels and lulian Niculescu, 2007).

The two well-known methods for approximating the delay terms in the dynamic system's model are the Padé approximant (Kumar and Chaudhary, 2017) and the Taylor series (Zwillinger, 2002). The first method uses rational components, while the second applies only polynomials.

According to (Insperger, 2015) if the order of the Taylor series expansion exceeds the order of the leading derivative by 2, the linearized system becomes unstable indifferent to the stability of the delayed system. The maximum order of the Taylor series can also be calculated by applying the Routh-Hurwitz stability criterion as explained in (Dorf and Bishop, 2001).

Another method for a delay-free approximation of a delayed system is the chain method (Győri, 1988) which uses a high-order dynamical system for approximation. The modified chain method described in (Krasznai et al., 2010) in some cases can be used for the uniform approximation of the solutions even on infinite interval.

In many applications, such as coordination-based control of multi-robot systems, the transient behavior of the agents is critical. In this work, we restrict ourselves to MAS where the communication delay is sufficiently small. Note that this smallness condition is naturally satisfied in many MAS application. Our aim is to develop an algorithm which combines numerical and symbolical computation methods to analyze the asymptotic behavior of a class of multi-agent systems with communication delay. The result of the algorithm is an order-preserving approximation of the multi-agent system with communication delay. The resulting model is a weighted MAS which has the same communication structure and steady states as the original system, and approximates well the transient behavior of the original MAS with communication delay.

Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ denote the set of natural numbers, the set of real numbers and the set of complex numbers. Let $\mathbb{R}^n$ be a set of column vectors with n real elements. Let $\mathbb{R}^{n\times n}$ be the set of matrices with $n\times n$ real elements, and $I_n \in \mathbb{R}^{n\times n}$ the identity matrix. Let $C(a_0, r)$ be the disc of center a_0 and radius r. Let $\|x\|$ be the l_1-norm of $x \in \mathbb{R}^n$ so that the induced matrix norm of $A \in \mathbb{R}^{n\times n}$ is given by $\|A\| = \max\limits_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$.

2. MODELS OF MULTI-AGENT SYSTEMS

In this paper, a MAS is considered with agents having single integrator dynamics. Thus the state space model of an agent becomes $\dot{x}_i(t) = u_i(t)$, where $x_i \in \mathbb{R}$ is the state of the i-th agent and $u_i \in \mathbb{R}$ is the input, $i = 1, 2, \ldots, n$.

A MAS has an underlaying communication graph, in which the vertex is an agent and the edge is a communication path (Trudeau, 1994), so the ith agent in the system is the vertex v_i. Let N_i be the set of neighbors of v_i, so that N_i contains all vertices that are connected to v_i.

Consensus algorithm

The consensus problem of a MAS is the procedure of gathering every state from the initial condition to a common steady-state. If the communication graph is connected, the consensus for an agent can be reached with the input (*consensus protocol*)

$$u_i(t) = \sum_{j\in N_i} \left(x_j(t) - x_i(t)\right). \tag{1}$$

The adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ of a graph with n nodes is defined as

$$a_{ij} := \begin{cases} 1, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

This matrix shows which vertices are neighbors in the graph.

The degree matrix $D = (d_{ij}) \in \mathbb{R}^{n\times n}$ of a graph with n nodes shows the number of neighbors for each vertex and can be defined as

$$d_{ij} := \begin{cases} \deg(v_i), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}, \tag{3}$$

where $\deg(v_i)$ denotes the degree, or the number of edges incident to vertex i.

With this notation the dynamics of the MAS with the consensus protocol (1) is given by

$$\underline{\dot{x}}(t) = -L\underline{x}(t),\ \underline{x}(0) = \underline{\xi}, \tag{4}$$

where L is the Laplacian matrix (Chaiken and Kleitman, 1978), which is constructed as $L = D - A$, D is the degree matrix, A is the adjacency matrix of the graph, $\underline{x} = (x_1\ x_2\ \ldots\ x_n)^T \in \mathbb{R}^n$ is the state vector consisting of the n states of the MAS and $\underline{\xi} \in \mathbb{R}^n$ is a constant vector.

The eigenvalues of $-L$ are given by the solutions of the characteristic polynomial of the state-space system (4), and they can be located in the Geršgorin circles (Varga, 2004)

$$C\bigg(-d_{kk}, \sum_{i=0,i\neq k}^{n} |a_{ik}|\bigg), \quad k = 1, 2, \ldots, n. \tag{5}$$

According to (Mesbahi and Egerstedt, 2010) the eigenvalues of a MAS consisting of n agents with a connected communication graph can be ordered as

$$0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n. \tag{6}$$

The steady states (equilibria) of the MAS $\underline{x}_{ss}$ are the elements of the null space of L. Since, by the definition of the Laplacian, $\sum_{j\in N_i} l_{ij} = 0$ according to (Lewis et al., 2014) for every solution $\underline{x}$ of (4) we have $\lim\limits_{t\to\infty} \underline{x}(t) = \underline{x}_{ss} = \frac{1}{n}\sum_{i=1}^{n} x_i(0)\mathbf{1}$, where $\mathbf{1} = (1\ 1\ \ldots\ 1)^T \in \mathbb{R}^n$.

Weighted MAS structures

In a weighted MAS, numbers (weights) are assigned to the edges, so the control input of an agent becomes

$$u_i(t) = \sum_{j\in N_i} w_{ij}\left(x_j(t) - x_i(t)\right), \tag{7}$$

where $w_{ij} > 0$. The dynamics of the MAS can be written in a matrix form similarly to relation (4) with $L_w = (l_{ij}) \in \mathbb{R}^{n\times n}$ given by

$$l_{ij} := \begin{cases} \sum\limits_{k\in N_i} w_{ik}, & \text{if } i = j \\ -w_{ij}, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}. \tag{8}$$

In this paper we will consider symmetric weights ($w_{ij} = w_{ji}$ for $1 \le i, j \le n$).

MAS with delayed communication

According to (Cheng-Lin Liu, 2017), and (Cepeda-Gomez and Olgac, 2011), the MAS with communication delay (further refereed to as Delayed Multi-Agent System (DMAS)) is

$$\underline{\dot{x}}(t) = -D\underline{x}(t) + A\underline{x}(t-\tau), \quad \underline{x}(\theta) = \underline{\xi}, \quad \theta \in [-\tau, 0], \tag{9}$$

where $\tau \geq 0$ is the constant delay which is present among neighbor agents and $\underline{\xi} \in \mathbb{R}^n$.

2.1 Reference tracking algorithm

In the case presented in the previous subsection, the consensus equilibrium is the weighted average of the agents' initial states, which is constant. If the state of the system has to converge to a reference vector, the stated control signal must be augmented with a reference tracking term so that

$$u_i(t) = \sum_{j\in N_i} \left(x_j(t) - x_i(t)\right) + k_{p_i}(x_{ri} - x_i(t)), \tag{10}$$

with a positive proportional gain k_{p_i} for the reference tracking of the ith agent, and x_{ri} is a given reference signal.

If we write the system in a matrix form, we get the relation

$$\underline{\dot{x}}(t) = -L\underline{x}(t) + K_p(\underline{x}_r - \underline{x}(t)), \tag{11}$$

where $K_p = \operatorname{diag}((k_{p1}\ k_{p2}\ \ldots\ k_{pn})) \in \mathbb{R}^{n\times n}$ is the diagonal proportional gain matrix, and $\underline{x}_r \in \mathbb{R}^n$ is the reference vector.

The eigenvalues of the system are located in the Geršgorin circles

$$C\bigg(-d_{kk} - k_{p_{kk}}, \sum_{i=0,i\neq k}^{n} |a_{ik}|\bigg), \quad k = 1, 2, \ldots, n, \tag{12}$$

We can see that the centers of the circles are shifted to the left by the proportional gains.

If a communication delay is introduced into the system, the state space representation is given by

$$\underline{\dot{x}}(t) = -(D + K_p)\underline{x}(t) + A\underline{x}(t-\tau) + K_p\underline{x}_r. \tag{13}$$

Note that the equilibria of the delayed MAS coincide with the equilibria of the system without delays ($\tau = 0$). The characteristic equation of the homogeneous part of the delayed system is given by

$$\det\left(\lambda I_n + D + K_p - Ae^{-\tau\lambda}\right) = 0. \tag{14}$$

This system in general has an infinite number of eigenvalues.

3. APPLIED METHODS

3.1 Differential equations with small delays

Consider a system of Delayed Differential Equation (DDE)

$$\dot{\underline{x}}(t) = F(t, \underline{x}_t), \tag{15}$$

where $F : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$, $\mathscr{C} \equiv C([-\tau, 0], \mathbb{R}^n)$ being the Banach space of continuous functions from $[-\tau, 0]$ into $\mathbb{R}^n$ equipped with the supremum norm and $\underline{x}_t \in \mathscr{C}$ is defined by $\underline{x}_t(\theta) = \underline{x}(t + \theta)$ for $\theta \in [-\tau, 0]$.

According to (Driver, 1976) if conditions

$$F : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n \text{ is continuous,} \tag{16}$$

$$\|F(t, 0)\| \leq He^{-t/\tau}, \qquad t \leq 0, \tag{17}$$

$$\|F(t, \phi) - F(t, \psi)\| \leq K\|\phi - \psi\|, \quad t \in \mathbb{R},\ \phi, \psi \in \mathscr{C}, \tag{18}$$

$$K\tau e < 1 \text{ (smallness condition)} \tag{19}$$

are satisfied, then for every solution x of (15) there exists a globally defined solution $\tilde{x} : \mathbb{R} \to \mathbb{R}^n$ of (15) satisfying the growth condition $\sup_{t\leq 0} \|\tilde{x}(t)\|e^{t/\tau} < \infty$ and such that

$$\|x(t) - \tilde{x}(t)\| \to 0 \qquad \text{exponentially as } t \to \infty.$$

The special solutions $\tilde{x}$ are uniquely determined by their values $\tilde{x}(0)$ and thus form an n parameter family. In the linear autonomous case they correspond to the eigensolutions generated by exactly n characteristic roots (counting multiplicities) which lie in the half plane $\operatorname{Re}\lambda > -1/\tau$, see (Arino and Pituk, 2001). For further related results, see (Győri and Pituk, 2005), and (Győri and Pituk, 2016).

3.2 The modified chain approximation method

A delay-free approximation model for the DDE

$$\dot{\underline{x}}(t) = A\underline{x}(t) + f(\underline{x}(t)) + g(\underline{x}(t - \tau)) + \underline{U} \tag{20}$$

has the form

$$\begin{aligned}
\dot{\underline{y}}_0(t) &= A\underline{y}_0(t) + f(\underline{y}_0(t)) + \frac{m}{\tau}I_n\underline{y}_m(t) + \underline{U} \\
\dot{\underline{y}}_1(t) &= g(\underline{y}_0(t)) - \frac{m}{\tau}I_n\underline{y}_1(t) \\
&\ \vdots \\
\dot{\underline{y}}_k(t) &= \frac{m}{\tau}I_n\underline{y}_{k-1}(t) - \frac{m}{\tau}I_n\underline{y}_k(t) + \underline{U}, \quad 2 \leq k \leq m,
\end{aligned} \tag{21}$$

where $A \in \mathbb{R}^{n\times n}$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are state dependent functions, and $U \in \mathbb{R}^n$. The output vector $\underline{z} = \underline{y}_0$ represents the approximation of the solution $\underline{x}$ of (20). More precisely, it was proven that there exists $c > 0$ such that $\sup_{t\geq 0} \|\underline{x}(t) - \underline{z}(t)\| \leq \frac{c}{m}$ independently of m. For details, see (Krasznai et al., 2010).

4. ASYMPTOTIC BEHAVIOR ANALYSIS METHOD

In this section, we propose an algorithm for the transient analysis of a given DMAS. The main steps are:

- Building up a higher order linear model with the help of the modified chain method.
- Finding the dominant eigenvalues of the system created with the modified chain method.
- Building up an order-preserving weighted MAS model from the dominant eigenvalues of the approximated system.

4.1 Higher order modified chain method approximation of MAS

In the case of the DMAS described by (13) the approximating chain system has the linear state-space form

$$\dot{\underline{Y}}(t) = G_A\underline{Y}(t) + G_B\underline{x}_r, \tag{22}$$

where $\underline{Y} \in \mathbb{R}^{mn}$ is the state vector, $G_A \in \mathbb{R}^{mn\times mn}$ and $G_B \in \mathbb{R}^{mn\times n}$ are the system matrices and $0_n \in \mathbb{R}^{n\times n}$ is the zero matrix. These matrices have the form

$$G_A = \begin{pmatrix} -(K_p + D) & 0_n & 0_n & \cdots & 0_n & \frac{m}{\tau}I_n \\ A & -\frac{m}{\tau}I_n & 0_n & \cdots & 0_n & 0_n \\ 0_n & \frac{m}{\tau}I_n & -\frac{m}{\tau}I_n & \cdots & 0_n & 0_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_n & 0_n & 0_n & \cdots & \frac{m}{\tau}I_n & -\frac{m}{\tau}I_n \end{pmatrix}$$

$$G_B = \left(K_p\ 0_n\ 0_n\ \ldots\ 0_n\right)^T, \tag{23}$$

where n is the number of agents and m is the number of approximating equations for DMAS. The solution $\underline{x}(t)$ of (13) is approximated by $\underline{z}(t) = G_C\underline{Y}(t)$, where $G_C \in \mathbb{R}^{n\times mn}$ is given by $G_C = (I_n\ 0_n\ \ldots\ 0_n)$.

4.2 Order preserving system approximation

The dominant eigenvalues of the chain approximation can be used to build an approximate model which preserves the order of the original one. In this step, we create an nth order weighted graph with the same communication structure as the original DMAS as shown in the relation

$$\frac{d\hat{\underline{x}}}{dt}(t) = -(L_w - K_p)\hat{\underline{x}}(t) + K_p\underline{x}_r, \tag{24}$$

where $\hat{\underline{x}} \in \mathbb{R}^n$ is the approximating state vector, and $\underline{x}_r$ is a constant reference vector. Its characteristic polynomial is

$$P_w(\lambda) = \det(\lambda I_n - (L_w + K_p))) = \lambda^n + \sum_{i=0}^{n-1} p_i(\underline{w})\lambda^i, \tag{25}$$

where $\underline{w}$ is the vector containing all the weights of the Laplacian matrix L_w defined by (8). The coefficients $p_i(\underline{w})$ are calculated as

$$\begin{aligned}
p_1(\underline{w}) &= \textstyle\sum_{i_1=1}^n w_{i_1} \\
p_2(\underline{w}) &= \textstyle\sum_{i_1=1}^n w_{i_1}(\sum_{i_2=i_1+1}^{n-1} w_{i_2}) \\
p_3(\underline{w}) &= \textstyle\sum_{i_1=1}^n w_{i_1}(\sum_{i_2=i_1+1}^{n-1} w_{i_2}(\sum_{i_3=i_2+1}^{n-2} w_{i_3})) \\
&\ \ etc.
\end{aligned}$$

The weights can be computed by solving the equation

$$P_w(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i), \tag{26}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the dominant eigenvalues of the system from the chain approximation. Note that the system (24) is constructed such that it has the same communication topology as the original DMAS (13), and their eigenvalues coincide with the n dominant eigenvalues of system (22) from the chain approximation.

The right-hand side of formula (26) contains a polynomial with positive coefficients since $\mathrm{Re}(\lambda_i) \leq 0$, $i = 1, \ldots, n$. The polynomial $P_w(\lambda)$ also has positive coefficients, since all the weights are positive.

Comparing the coefficients in equations (25) and (26), we obtain a system of n nonlinear equations. If we consider only the consensus protocol, this system contains $n-1$ equations since both polynomials have $\lambda = 0$ as a root.

4.3 Algorithm to obtain the order-preserving approximation of DMAS

Algorithm 1: Algorithm for the transient analysis of a MAS with communication delay.

Input : τ real, A, D, K_p matrices
Output: L_w matrix, $\underline{\hat{x}}(t)$ trajectories

1 Compute the Lipschitz constant $K = \|D + K_p\| + \|A\|$
2 **if** $\tau \geq \frac{1}{Ke}$ **then**
3   The method can only be applied in case of smaller delays. **return**
4 **end**
5 Construct the model based on the modified chain method according to (22)
6 Construct the characteristic equation of the system by applying the LU factorization symbolically with partial pivoting
7 Find dominant eigenvalues λ_i with $\mathrm{Re}(\lambda_i) \in (-\frac{1}{\tau}, 0]$ interval using numerical methods
8 Construct a general weighted MAS with n nodes according to (8).
9 Construct symbolically the characteristic polynomial P_w of the weighted graph according to (25).
10 Calculate the coefficients of the characteristic equation so that the eigenvalues are the dominant ones from the approximated model according to (26).
11 Calculate the weights $\underline{w}$ from the coefficients.
12 Generate the trajectories $\underline{\hat{x}}(t)$ using (24).
13 **return** $L_w, \underline{\hat{x}}(t)$

The algorithm inputs are the communication delay τ, the adjacency matrix A, degree matrix D, and gain K_p (if applicable) of the DMAS, and returns the approximating linear system matrix L_w, the weight vector $\underline{w}$ and the trajectories $\underline{\hat{x}}(t)$ of the weighted delay-free MAS obtained by the order preserving approximation.

Detailed steps of the devised algorithm:

(1) Define $K = \|D + K_p\| + \|A\|$, and check the smallness condition (19) , according to Section 3.1. Conditions (16) and (17) are always satisfied for DMAS with $H = e^{-1}\|K_p\|\|\underline{\xi}\|$. If the delay τ is greater than $(Ke)^{-1}$ this method cannot be applied.

(5) Construct the system (22) symbolically based on the modified chain approximation method with matrices given by (23).

(6) Create the LU factorization with partial pivoting of the term $(I_{mn}\lambda - G_A)$ according to $Q(I_{mn}\lambda - G_A) = LU$, where $L, U, Q \in \mathbb{R}^{mn \times mn}$ are the lower triangle, upper triangle and the permutation matrices respectively. Since the chain method approximation results in a sparse matrix configuration, the decomposition has a complexity of *O(n)* according to (Datta, 2010).
The characteristic equation of the chain approximated system can be calculated symbolically with the help of the above mentioned LUP factorization as $P(\lambda) = \det(Q^{-1})\det(L(\lambda))\det(U(\lambda))$. Thus the characteristic equation can be computed by multiplying the diagonal terms of the $L(\lambda)$ and $U(\lambda)$ matrices.

(7) Find all the dominant eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the chain approximating system for which $\mathrm{Re}(\lambda_i) \in (-\frac{1}{\tau}, 0]$ using numerical methods. One real or complex pair of roots can be found with the help of Bairstow's method (Press et al., 2007). After finding the root(s), the characteristic polynomial can be divided according to Horner's method to eliminate the already found roots. This iteration is continued until all n eigenvalues are found.

(8,9) The order-preserving approximating system is created according to Section 4.2, for which the eigenvalues are in the circle $C\left(0, \frac{1}{\tau}\right)$.
The dominant eigenvalues of the order preserved approximating system converge to the dominant eigenvalues of the delayed system, see (Yanushevskij, 1978).
The characteristic equation $P_w(\lambda)$ is computed symbolically using the Gaussian elimination method with complexity $O(n^3)$.

(10,11) The weights of the order preserving approximator are calculated by comparing the equations (25) and (26). This creates an underdetermined nonlinear system of equations which can be solved numerically.

5. CASE STUDIES

In this section, we apply the algorithm to study the asymptotic behavior of two DMASs. The algorithm is written and tested in MATLAB® using the Symbolic Math Toolbox.

5.1 Consensus problem with communication delay

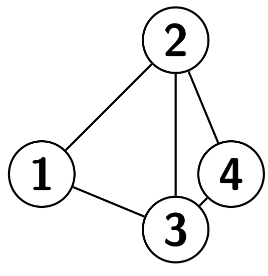

Fig. 1. The communication topology of the first MAS.

Consider the DMAS structure shown in Figure 1 with $\tau = 0.03s$ delay. The delayed model is written by (9) with

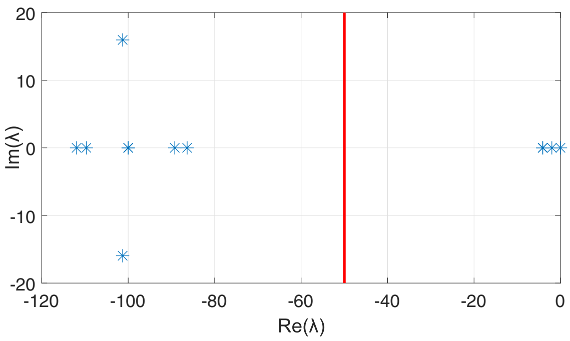

Fig. 2. The eigenvalues of the approximated system based on the chain method. The vertical line shows the $-\frac{1}{\tau}$ boundary.

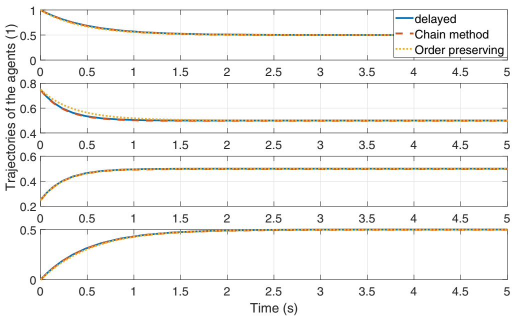

Fig. 3. Comparison between the original DMAS, the result of the chain approximation, and the final order preserving approximation.

$D = \mathrm{diag}(2,3,3,2)$, $A = \begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$, so that the Laplacian matrix is $L = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$.

Condition (18) is fulfilled with $K = \|D\| + \|A\| = 6$. Thus the algorithm works with delays $\tau \in [0, \frac{1}{6e})$.

Figure 2 shows the eigenvalues of the system obtained from the chain approximation. Four eigenvalues $(-4, -4, -2, 0)$ are found in the $(-\frac{1}{\tau}, 0]$ interval.

The `sym2poly` Matlab function was used to convert the symbolically calculated characteristic equation of the weighted to an array containing the coefficients of the polynomial equation. The numeric solution of the nonlinear equation from step 10 of Algorithm 1 can be found with the help of the function `fsolve` algorithm. The default *trust-region-reflective* method is not suitable for this purpose, since the system is a nonlinear, underdetermined system, therefore the *Levenberg-Marquardt* method was used.

The weights for the weighted graph are $w_{12} = 1.3448$, $w_{13} = 1.1366$, $w_{23} = 0.8539$, $w_{24} = 0.7576$, $w_{34} = 1.0848$.

The comparison of the trajectories of the original system, the chain approximation and the order preserving approximation can be seen in Figure 3.

5.2 Reference tracking with communication delay

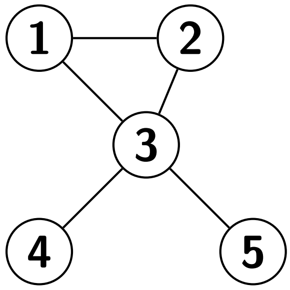

Fig. 4. The communication topology of the second MAS.

In case of the DMAS structure shown on Figure 4, with a time delay $\tau = 0.015$ the model is writtenby (13) with $D = \mathrm{diag}(2,2,4,1,1)$, $A = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$, so that the Laplacian matrix is $L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}$, and $K_p = \mathrm{diag}((1\ 1\ 3\ 2\ 2))$ gain. The reference state is $\underline{x}_r = 0.8 \cdot \mathbf{1}$.

The Lischitz constant is $K = \|D + K_p\| + \|A\| = 11$. Thus the algorithm works with delays $\tau \in [0, \frac{1}{11e})$.

Figure 5 shows the eigenvalues of the system after the chain approximation. The five eigenvalues found are $(-8.2393, -4.1547, -3.0000, -2.6768, -1.4238)$.

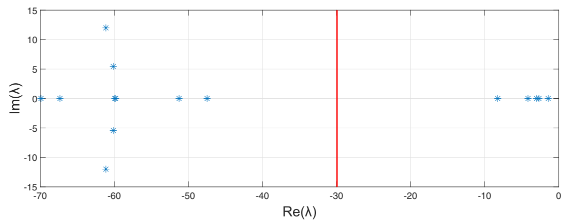

Fig. 5. The eigenvalues of the approximated system based on the chain method. The vertical line shows the $-\frac{1}{\tau}$ boundary.

The weights for the weighted graph are $w_{12} = 0.9269$, $w_{13} = 1.2104$, $w_{23} = 1.2104$, $w_{35} = 1.0606$, $w_{34} = 0.7040$.

The comparison of the trajectories of the original system, the chain approximation and the order preserving approximation for this case can be seen in Figure 6.

6. CONCLUSION

A new algorithm was introduced based on which an order preserving approximation model for a class of MAS with small delay can be determined. The algorithm combines numerical and symbolical computation methods that are available in commercial mathematical software. The obtained weighted delay-free MAS has the same order, communication structure and steady-state as the original

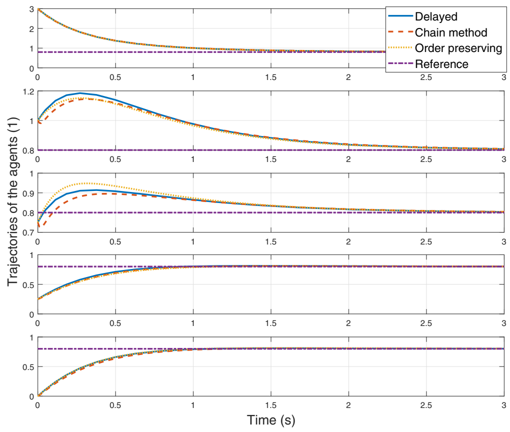

Fig. 6. Comparison between the original DMAS, the result of the chain approximation, and the final order preserving approximation.

DMAS. Moreover, it approximates correctly the transient behavior of the DMAS. The presented numerical examples confirm the applicability of the proposed asymptotic behavior analysis method. In our case studies we restricted ourselves to simple DMAS. In the future we plan to test whether the presented algorithm is suitable for complex MAS. Furthermore, we plan to compare our algorithm with the existing methods, see, e.g. (Jarlebring, 2008), (Olfati-Saber and Murray, 2004), (Breda et al., 2014).

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