Approximate Lyapunov matrices for time-delay systems

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Abstract: A numerical scheme to approximate delay Lyapunov matrices for exponentially stable linear time-delay systems with distributed delay is given.

Keywords: time-delay systems, delay Lyapunov matrices.

1. INTRODUCTION

The construction of quadratic Lyapunov-Krasovskii functionals with a prescribed time derivative has been initiated by Repin [1965]. A brief historic overview of the main contributions in the area can be found in Kharitonov [2013], see Section 2.13. At the very beginning it became clear that the functionals may be used not only to check stability of a time-delay system, but constitute a powerful tool for analysis and design of time-delay systems Marshall et al. [1992], Jarlebring et al. [2011], Kharitonov and Zhabko [2003], Ochoa et al. [2013]. In applications the positive effect of the functionals depends on the existence of reliable numerical procedures for the computation of delay Lyapunov matrices. There exists a standard procedure for the computation of the matrices for systems with delays multiple to a basic one. In this case the computation is reduced to evaluation of a solution of two-point boundary value problem for an auxiliary system of ordinary matrix differential equations Kharitonov [2013]. The computational scheme has been extended to the case of distributed delay systems with special kernels, see Kharitonov [2006].

In this paper a new approximation scheme for the computation of delay Lyapunov matrices for the exponentially stable systems with distributed delay is presented. An upper estimate of the approximation error is given.

The paper is organized as follows. In Section 2 a time-delay systems studied in the paper is described. Then an approximate system with delays multiple to a basic one is introduced. Section 3 is dedicated to the evaluation of the difference of the fundamental matrices for the original and the approximate systems. In Section 4 an upper estimate of the approximation error of the delay Lyapunov matrix is given. An illustrative example ends the section.

2. SYSTEM DESCRIPTION

Given a time-delay system of the form

$$\frac{dx(t)}{dt} = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)d\theta, \quad (1)$$

where A_0 and A_1 are real $n \times n$ matrices, $G(\theta)$ is a continuous matrix valued function, h is a positive delay. The system state x_t is defined as follows

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-h, 0].$$

Let N be a natural number, we approximate the distributed delay term in (1) by a finite sum of the form

$$\sum_{j=0}^{N-1} Q_j x(t - j\delta),$$

where $\delta = \frac{h}{N}$ and

$$Q_j = \int_{-(j+1)\delta}^{-j\delta} G(\theta)d\theta, \quad j = 0, 1, ..., N-1;$$

and define a new time-delay system as follows

$$\frac{dz(t)}{dt} = A_0z(t) + A_1z(t-h) + \sum_{j=0}^{N-1} Q_j z(t - j\delta). \quad (2)$$

All delays of the new system are multiple to δ.

Remark 1. If system (1) is exponentially stable then for sufficiently large N system (2) is also exponentially stable, see Kharitonov [2014].

In the following we assume that systems (1) and (2) are exponentially stable.

Denote by $K_0(t)$ and $K_1(t)$ fundamental matrices of the systems, see Bellman and Cooke [1963]. By definition matrix $K_0(t) = 0_{n\times n}$ for $t < 0$, $K_0(0) = I$, and for $t \geq 0$ the matrix satisfies the equation

$$\frac{dK_0(t)}{dt} = A_0K_0(t) + A_1K_0(t-h) + \int_{-h}^{0} G(\theta)K_0(t+\theta)d\theta.$$

Similarly, matrix $K_1(t) = 0_{n\times n}$ for $t < 0$, $K_1(0) = I$, and

$$\frac{dK_1(t)}{dt} = A_0K_1(t) + A_1K_1(t-h) + \sum_{j=0}^{N-1} Q_jK_1(t-j\delta), \quad t \geq 0.$$

Exponential stability of the systems implies that that there exist $\gamma \geq 1$ and $\sigma > 0$ such that the fundamental matrices satisfy the inequalities

$$\|K_j(t)\| \leq \gamma e^{-\sigma t}, \quad j = 0, 1. \qquad (3)$$

3. LYAPUNOV MATRICES

Since systems (1) and (2) are exponentially stable the corresponding delay Lyapunov matrices of the systems can be presented in the form

$$U_0(\tau) = \int_0^\infty K_0^T(t)WK_0(t+\tau)dt,$$

and

$$U_1(\tau) = \int_0^\infty K_1^T(t)WK_1(t+\tau)dt,$$

respectively.

Remark 2. Kharitonov [2013] It is known that system (1) admits a unique Lyapunov matrix associated with a given symmetric matrix W if and only if the the system satisfies the Lyapunov condition, i.e., the system has no an eigenvalue s_0 such that $-s_0$ is also an eigenvalue of the system. The same is valid for system (2).

Define the difference

$$\begin{aligned}\Delta U(\tau) &= U_1(\tau) - U_0(\tau) \qquad (4)\\ &= \int_0^\infty K_1^T(t)W\Delta K(t+\tau)dt \\ &\quad + \int_0^\infty [\Delta K(t)]^T WK_0(t+\tau)dt,\end{aligned}$$

where $\Delta K(t) = K_1(t) - K_0(t)$.

4. ESTIMATION OF THE NORM $\|\Delta K\|$

The difference

$$\Delta K(t) = K_1(t) - K_0(t)$$

satisfies the equation

$$\frac{d\Delta K(t)}{dt} = A_0\Delta K(t) + A_1\Delta K(t-h) + \int_{-h}^{0} G(\theta)\Delta K(t+\theta)d\theta + F(t), \quad t \geq 0,$$

where

$$F(t) = \sum_{j=0}^{N-1} \int_{-(j+1)\delta}^{-j\delta} G(\theta)\left[K_1(t-j\delta) - K_1(t+\theta)\right]d\theta.$$

The standard variation-of-constants formula provides the following expression for the difference

$$\begin{aligned}\Delta K(t) &= K_0(t)\Delta K(0) + \int_{-h}^{0} K_0(t-h-\theta)A_1\Delta K(\theta)d\theta \\ &\quad + \int_{-h}^{0}\left(\int_{-h}^{\theta} K_0(t-\theta-\xi)G(\xi)d\xi\right)\Delta K(\theta)d\theta \\ &\quad + \int_0^t K_0(t-\xi)F(\xi)d\xi, \quad t \geq 0.\end{aligned}$$

As $\Delta K(t) = 0_{n\times n}$ for $t \in [-h, 0]$, the formula taks the form

$$\Delta K(t) = \int_0^t K_0(t-\xi)F(\xi)d\xi$$

and after direct computations we conclude that

$$\Delta K(t) = \sum_{j=0}^{N-1} R_j(t), \quad t \geq 0,$$

where

$$R_j(t) = \int_{-j\delta}^{t-j\delta} K_0(t-\lambda-j\delta) \times \int_{-\delta}^{0} G(\mu - j\delta)\left[K_1(\lambda) - K_1(\lambda+\mu)\right]d\mu d\lambda.$$

Lemma 1. The following upper estimate holds for matrix $R_j(t)$,

$$\|R_j(t)\| \leq \frac{1}{N^2}h^2\gamma^2M(2+at)e^{-\sigma(t-h)}, \quad t \geq 0,$$

where

$$M = \max_{\theta\in[-h,0]} \|G(\theta)\|,$$

and

$$a = \|A_0\| + e^{\sigma h}(\|A_1\| + hM).$$

Proof. First, we observe that for $t < j\delta$ matrices $K_1(\lambda) = K_1(\lambda+\mu) = 0_{n\times n}$, and

$$R_j(t) = 0_{n\times n}, \quad t \in [0, j\delta). \qquad (5)$$

For $t \geq j\delta$ the matrix can be written as

$$R_j(t) = \int_0^{t-j\delta} K_0(t-\lambda-j\delta) \times \int_{-\delta}^{0} G(\mu - j\delta)\left[K_1(\lambda) - K_1(\lambda+\mu)\right]d\mu d\lambda.$$

Applying inequalities (3) we obtain that for $t \in [j\delta, (j+1)\delta]$

$$\|R_j(t)\| \leq \frac{2}{N^2}h^2\gamma^2Me^{-\sigma(t-h)}. \qquad (6)$$

In the case when $t \geq (j+1)\delta$ the term $R_j(t)$ may be written in the form

$$R_j(t) = \int_0^{\delta} K_0(t-\lambda-j\delta)$$
$$\times \int_{-\delta}^{0} G(\mu - j\delta)\left[K_1(\lambda) - K_1(\lambda+\mu)\right] d\mu d\lambda$$
$$+ \int_{\delta}^{t-j\delta} K_0(t-\lambda-j\delta)$$
$$\times \int_{-\delta}^{0} G(\mu - j\delta)\left[K_1(\lambda) - K_1(\lambda+\mu)\right] d\mu d\lambda$$

The first summand on the right hand side of the preceding expression,

$$R_{j1}(t) = \int_0^{\delta} K_0(t-\lambda-j\delta)$$
$$\times \int_{-\delta}^{0} G(\mu - j\delta)\left[K_1(\lambda) - K_1(\lambda+\mu)\right] d\mu d\lambda,$$

is such that

$$\|R_{j1}(t)\| \leq \frac{2}{N^2} h^2\gamma^2 M e^{-\sigma(t-h)}.$$

In the second summand,

$$R_{j2}(t) = \int_{\delta}^{t-j\delta} K_0(t-\lambda-j\delta)$$
$$\times \int_{-\delta}^{0} G(\mu - j\delta)\left[K_1(\lambda) - K_1(\lambda+\mu)\right] d\mu d\lambda,$$

the value $\lambda \geq \delta$ and $\mu \in [-\delta, 0]$, therefore the difference $K_1(\lambda) - K_1(\lambda+\mu)$ can be presented as follows

$$K_1(\lambda) - K_1(\lambda+\mu) = \int_{\lambda+\mu}^{\lambda} \frac{dK_1(\xi)}{d\xi} d\xi$$
$$= \int_{\lambda+\mu}^{\lambda} \left[A_0 K_1(\xi) + A_1 K_1(\xi - h) + \sum_{i=0}^{N-1} Q_i K_1(\xi - i\delta)\right] d\xi.$$

This implies that

$$\|K_1(\lambda) - K_1(\lambda+\mu)\| \leq \gamma \left[\|A_0\|\right.$$
$$\left. + e^{\sigma h}\left(\|A_1\| + hM\right)\right] \int_{\lambda+\mu}^{\lambda} e^{-\sigma\xi} d\xi$$
$$\leq \gamma a \delta e^{-\sigma(\lambda-\delta)}.$$

Now we estimate the second summand,

$$\|R_{j2}(t)\| \leq \delta\gamma^2 a e^{-\sigma(t-(j+1)\delta)}$$
$$\times \int_{\delta}^{t-j\delta} \int_{-\delta}^{0} \|G(\mu - j\delta)\| \, d\mu d\lambda$$
$$\leq \frac{1}{N^2} h^2\gamma^2 a e^{-\sigma(t-h)} M t, \quad t \geq (j+1)\,\delta.$$

And, finally,

$$\|R_j(t)\| \leq 2\delta^2\gamma^2 M e^{-\sigma(t-h)} + \delta^2\gamma^2 a M t e^{-\sigma(t-h)} \quad (7)$$
$$= \frac{1}{N^2} h^2\gamma^2 M(2+at) e^{-\sigma(t-h)}, \quad t \geq (j+1)\,\delta.$$

Inequalities (5)-(7) justify the lemma statement.

Corollary 1. The difference $\Delta K(t)$ is such that the following inequality holds

$$\|\Delta K(t)\| \leq \frac{1}{N} h^2\gamma^2 M(2+at) e^{-\sigma(t-h)}.$$

Proof. It follows from

$$\Delta K(t) = \sum_{j=0}^{N-1} R_j(t)$$

that

$$\|\Delta K(t)\| \leq \sum_{j=0}^{N-1} \|R_j(t)\|$$
$$\leq \frac{1}{N} h^2\gamma^2 M(2+at) e^{-\sigma(t-h)}.$$

4.1 Estimation of $\|\Delta U(\tau)\|$

We now address the difference (4).

Theorem 1. The difference $\Delta U(\tau) = U_1(\tau) - U_0(\tau)$ admits an upper estimate of the form

$$\|\Delta U(\tau)\| \leq \frac{\alpha}{N} e^{-\sigma(\tau-h)} \left(\frac{4 + a\,|\tau|}{2\sigma} + \frac{a}{2\sigma^2}\right),$$

where

$$\alpha = h^2\gamma^3 M \|W\|.$$

Proof. It is enough to study the case when $\tau \geq 0$. It follows from (4) that

$$\|\Delta U(\tau)\| \leq \|W\| \int_0^{\infty} \|K_1(t)\| \, \|\Delta K(t+\tau)\| \, dt$$
$$+ \|W\| \int_0^{\infty} \|\Delta K(t)\| \, \|K_0(t+\tau)\| \, dt.$$

Applying Corollary 1 and inequalities (3) we arrive at the desired result

$$\|\Delta U(\tau)\| \leq \frac{1}{N} h^2 \gamma^3 M \|W\| e^{-\sigma(\tau-h)}$$
$$\times \int_0^\infty (2 + at + a\tau) e^{-2\sigma t} dt$$
$$+ \frac{1}{N} h^2 \gamma^3 M \|W\| e^{-\sigma(\tau-h)} \int_0^\infty (2 + at) e^{-2\sigma t} dt$$
$$= \frac{1}{N} h^2 \gamma^3 M \|W\| e^{-\sigma(\tau-h)} \left(\frac{4 + a\tau}{2\sigma} + \frac{a}{2\sigma^2} \right).$$

Corollary 2. For $\tau \in [-h, h]$ the difference (4) is such that the following inequality holds

$$\|U_1(\tau) - U_0(\tau)\| \leq \frac{\alpha_1}{N},$$

where

$$\alpha_1 = \alpha e^{\sigma h} \left(\frac{4 + ah}{2\sigma} + \frac{a}{2\sigma^2} \right).$$

Example 1. Consider the following system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) \tag{8}$$
$$-0.2 \int_{-1}^{0} \left[\theta e^{-\theta} B_0 + e^{-\theta} B_1 + B_2 \right] x(t+\theta) d\theta,$$

where

$$A_0 = \begin{pmatrix} -3 & 1 \\ 0 & -4 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 2 & -1 \\ 4 & 0 \end{pmatrix},$$
$$B_0 = \begin{pmatrix} 4 & -1 \\ 5 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -2 & 5 \\ 5 & 5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}.$$

Define approximate systems of the form (2) with $N = 5$ and $N = 10$. The original distributed delay system and the approximate ones are exponentially stable. The fundamental matrices of the systems satisfies inequality (3) with $\sigma = 0.4$ and $\gamma = 31.4$. Let $W = I_2$, the corresponding Lyapunov matrix $U_0(\tau)$ for system (8) has been computed with algorithm provided in Aliseyko [2017]. Computation of Lyapunov matrices for approximate systems has been performed with a standard scheme for systems with delays multiple to a basic one. On Figure 1 the components of matrces $U_0(\tau)$ and $U_1(\tau)$ for $N = 5$ are presented.

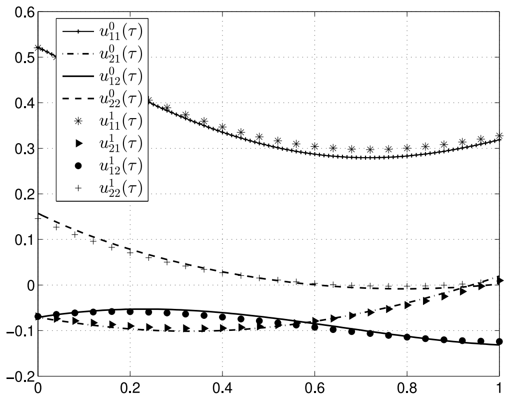

Fig. 1. $U_0(\tau)$ and $U_1(\tau)$, $N = 5$

On Figure 2 the components of matrces $U_0(\tau)$ and $U_1(\tau)$ for $N = 10$ are given.

It follows from Figure 3, where the graph of the norm $\|\Delta U(\tau)\|$ for $N = 5$ is presented, that this norm is bounded by 0.0193.

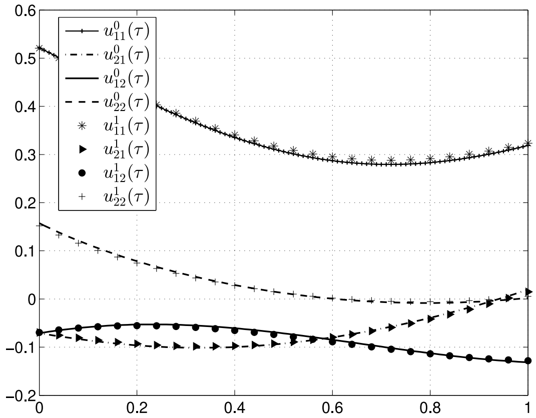

Fig. 2. $U_0(\tau)$ and $U_1(\tau)$, $N = 10$

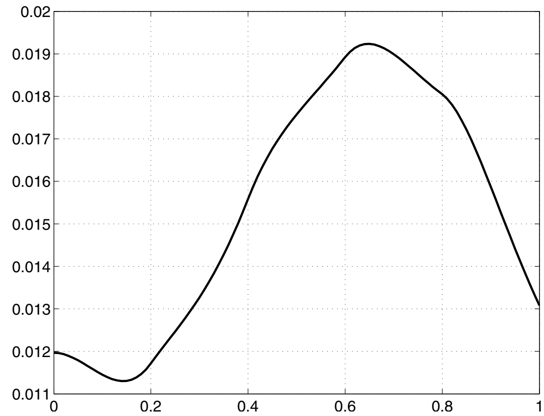

Fig. 3. $\|U_0(\tau) - U_1(\tau)\|, N = 5$

The corresponding graph for $N = 10$ is given on Figure 4, here the norm is bounded by 0.0093.

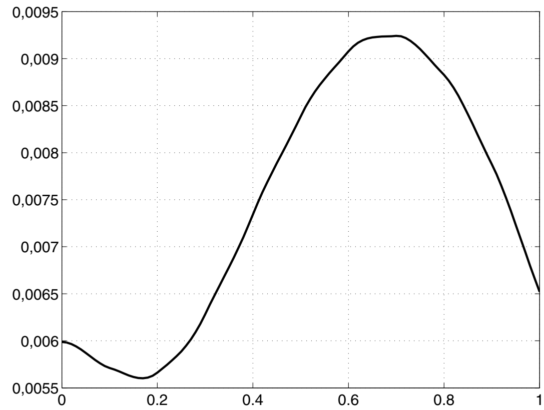

Fig. 4. $\|U_0(\tau) - U_1(\tau)\|, N = 10$

I extend my thanks to my MS student Alexey Aliseyko who performed all computations of the example.

5. CONCLUSION

A numerical scheme for approximation of delay Lyapunov matrices for time delay systems with distributed delay is given. The original system with distributed delay is approximated by a system, where the distributed delay terms are replaced by finite integral sums, and then a corresponding delay Lyapunov matrix for the new system is considered as the desired approximation of the delay Lyapunov matrix for the original system. It is shown that the approximation error tends to zero as the number of partition points in the finite integral sum is increasing. Only the case of exponentially system has been analyzed since our study is essentially based on the integral expression for delay Lyapunov matrices. In the future research we hope to remove this limitation constraint.

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