Some Remarks on the Asymptotic Behavior for Quasipolynomials with Two Delays

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Abstract: This paper proposes an analytical method to characterize the behavior of critical multiple roots for quasi-polynomials with two delays. The proposed approach is based on the Weierstrass polynomial, that is employed as a tool to analyze the stability behavior of such characteristic roots with respect to small variations on the delay parameters. A numerical example illustrates the proposed results.

Keywords: Quasi-polynomial, Retarded systems, Newton Diagram, Weierstrass Polynomial.

1. INTRODUCTION

The stability analysis of linear time-invariant (LTI) systems with time-delay have been studied for a long time, and there exists an abundant literature covering this subject, see, for example, Gu et al. (2003), Michiels and Niculescu (2014) and the references therein. Even though delays have been associated with "undesired" behaviors (as, for examples, oscillations, bandwidth sensitivity), recently has been shown that in some situations the presence of delays may induce stability, where the classical example was presented in Abdallah et al. (1993), where a simple oscillator is controlled by one delay "block" (gain, delay), with positive gains and extremely small delay values. As discussed in Michiels and Niculescu (2007), such a property opens an interesting perspective in using *delays* as *control parameters* in some situations, as, for example, in Niculescu and Michiels (2004) (stabilizing chains of integrators by using delays), Kharitonov et al. (2005) (multiple delay blocks), and Mazenc et al. (2003) (bounded input, single delay). Nonetheless, the approach appears *conservative* in other cases; see, for instance, Michiels and Niculescu (2007); Sipahi et al. (2011).

In the works of Chen et al. (2010b), the above observations have been explored in detail for a general *retarded* LTI delay system with multiple commensurate delays. Specifically, they have first fully characterized the stability properties of such systems by proposing conditions to find the set of critical delay values, at which the system's characteristic quasi-polynomial has critical zeros on the imaginary axis. Secondly, considering the delay as a variable parameter and by adopting an operator based-approach (see, for instance, Chen et al. (2010a)) they have expanded the solutions of the quasi-polynomial in terms of a Taylor (or Puiseux) series, allowing them to analyze the behavior of the solutions as the delay varies around a critical delay value.

It is well recognized (Chen et al. (2010b)) that even in the case of a fixed delay, the testing of stability for a time-delay system is not a simple task. Furthermore, it is well known that delay systems (and in consequence, quasi-polynomials) have always infinitely many solutions (see, for further details, Gu et al. (2003) and the references therein), however in general, we will only be interested in analyzing the behavior of a critical zero of finite multiplicity.

Multiple delays can also be presented in a more general form, as non-commensurate delays, that is, all delays are assuming to be independent of each other. In this case, the stability of the related quasi-polynomial becomes more complex and is less studied. For the case of solutions of multiplicity two, in Irofti et al. (2018) the authors have proposed two sectors to study the behavior of these roots when delays are subject to small deviations and restricted to such sectors. In this vein, it is worth to mention that unlike the case of a single parameter, in the multiparameter case there exist some singular and unexpected behaviors (see, the motivating examples section) which have to be taken into account (see, for instance Monforte and Kauers (2013)), in order that the problem is well-posed. In other words, the Puiseux type arguments cannot be extended straightforwardly from one parameter to multiparameter case.

Based on the above arguments, the main goal of this paper is two-fold. First, give conditions that guarantee the existence of a convergent Puiseux (or Taylor) series solution around roots of multiplicity $m > 1$. Second, extend the use of the well known Newton diagram to the case of two parameters.

The remaining part of the paper is organized as follows: Section II introduces some preliminary results, motivating examples and the problem formulation. Section III is devoted to the main results; specifically we present a method to compute the Weierstrass Polynomial, and an algorithm to find Newton polygon. Furthermore, necessary conditions to obtain Generalized Puiseux series are presented. Finally, section IV a numerical example illustrates the proposed results. The contribution ends with some concluding remarks.

Notations: In the sequel, the following notations will be adopted: $\mathbb{C}$ is the set of complex numbers, $\boldsymbol{i} := \sqrt{-1}$. Next, $\mathbb{R}_+$ denotes the set of positive real values. The *order* of a power series $f(x,y) = \sum a_{i,j}x^iy^j$ will be denoted by $\operatorname{ord}(f)$ and defined as the smallest number $n = i + j$ such that $a_{i,j} \neq 0$. The ring of complex formal power series is denote by $\mathbb{C}[[x]]$, with subring $\mathbb{C}\{x\}$ of convergent power series. Finally, given two polynomials $f(z) = \sum_{j=0}^{n} a_{n-j}z^j$ and $g(z) = \sum_{j=0}^{m} b_{m-j}z^j$, the *resultant* of f, g is defined as the determinant of *Sylvester matrix* as follows

$$\mathcal{R}(f,g) := \det \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & \\ & a_0 & a_1 & \cdots & \cdots & a_n & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \cdots & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & b_m & & & \\ & b_0 & b_1 & \cdots & \cdots & b_m & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \cdots & \cdots & b_m \end{bmatrix},$$

with n rows of a_i and m rows of b_i.

2. PRELIMINARIES

2.1 Retarded Linear Time-Invariant Systems

Consider a retarded LTI system with h_a-delays τ_k, as

$$\dot{x}(t) = A_0x(t) + \sum_{k=1}^{h_a} A_kx(t-\tau_k), \qquad \tau_k \geq 0, \quad (1)$$

with characteristic function given by the quasi-polynomial:

$$f(s,\tau) = \sum_{k=0}^{h_a} p_k(s)e^{-\tau_k s}, \qquad \tau_k \geq 0, \quad (2)$$

where the polynomials p_k are given by

$$p_0(s) = s^n + \sum_{\ell=0}^{n-1} a_{0\ell}s^\ell, \quad p_k(s) = \sum_{\ell=0}^{n-1} a_{k\ell}s^\ell, \quad k = 1,\ldots,h_a.$$

In order to perform an asymptotic behavior analysis of multiple imaginary roots, we will make use of the following results and definitions.

2.2 Local Representation of Analytic Functions

It it possible to reduce the analytic properties of $f(x,y)$ to algebraic ones. To this purpose, let us consider the following result (for further details see Mailybaev and Grigoryan (2001)).

Theorem 1. (Weierstrass Preparation Theorem). Let $f(z,\boldsymbol{x})$ be an analytic function vanishing at the singular point $z_0 \in \mathbb{C}$, $\boldsymbol{x}_0 \in \mathbb{C}^n$, where $z = z_0$ is an $m-$multiple root of the equation $f(z,\boldsymbol{x}) = 0$, i.e.,

$$f(z_0,\boldsymbol{x}_0) = \frac{\partial f}{\partial z} = \cdots = \frac{\partial^{m-1} f}{\partial z^{m-1}} = 0, \quad \frac{\partial^m f}{\partial z^m} \neq 0.$$

where derivatives are evaluated at $(z_0,\boldsymbol{x}_0)$. Then, there exist a neighborhood $U_0 \subset \mathbb{C}^{n+1}$ of the point $(z_0,\boldsymbol{x}_0) \in \mathbb{C}^{n+1}$ in which the function $f(z,\boldsymbol{x})$ can be expressed as

$$f(z,\boldsymbol{x}) = W(z,\boldsymbol{x})\,b(z,\boldsymbol{x}), \quad (3)$$

where $W(z,\boldsymbol{x})$ is given by

$$(z-z_0)^m + w_{m-1}(\boldsymbol{x})(z-z_0)^{m-1} + \cdots + w_0(\boldsymbol{x}),$$

and $w_0(\boldsymbol{x}),\ldots,w_{m-1}(\boldsymbol{x})$, $b(z,\boldsymbol{x})$ are analytic functions uniquely defined by the function $f(z,\boldsymbol{x})$, and $w_i(\boldsymbol{x}_0) = 0$, $b(z_0,\boldsymbol{x}_0) \neq 0$.

Remark 2. The holomorphic function

$$W(z,\boldsymbol{x}) = z^m + w_{m-1}(\boldsymbol{x})z^{m-1} + \cdots + w_0(\boldsymbol{x}), \quad (4)$$

is known as the *Weierstrass polynomial* (for further details on Weierstrass polynomials, see, for instance, Wall (2004)).

Remark 3. It can be seen from Theorem 1, that since $b(z,\boldsymbol{x})$ is an holomorphic non vanishing function at $(z_0,\boldsymbol{x}_0)$, then, there must exist some neighborhood $U \subset \mathbb{C}^n$ at which $b(z,\boldsymbol{x})$ preserves the same property. Hence, based on this observation we can ensure that the roots behavior of a given quasi-polynomial f in the neighborhood U will be completely described by the roots behavior of $W(z,\boldsymbol{x})$.

2.3 Newton Diagram Method

It is well known that solutions of the equation $f(x,y) = 0$ with $x,y \in \mathbb{C}$ can be computed term by term by means of the *Newton Diagram Method.* Thus, in order to use such a procedure, let us introduce the following notation (for more details, see, for instance, Vainberg and Trenogin (1974)). Let $f(x,y)$ be a *pseudo-polynomial* in y, i.e.,

$$f(x,y) = \sum_{k=0}^{n} a_k(x)y^k, \quad (5)$$

where the corresponding coefficients are given by,

$$a_k(x) = x^{\rho_k}\sum_{r=0}^{\infty} a_{rk}x^{r/q}, \quad (6)$$

$a_{rk} \in \mathbb{C}$, x and y are complex variables, ρ_k are non-negative rational numbers, q is an arbitrary natural number, $a_n(x) \not\equiv 0$, and $a_0(x) \not\equiv 0$.

Since by simple translation, any point on a curve can be moved to the origin, we will consider expansions of the solution of (5) $f(x,y) = 0$ around the origin, in the following form

$$y(x) = y_{\epsilon_1}x^{\epsilon_1} + y_{\epsilon_2}x^{\epsilon_2} + y_{\epsilon_3}x^{\epsilon_3} + \cdots, \quad (7)$$

where $\epsilon_1 < \epsilon_2 < \epsilon_3 < \cdots$ and $y_{\epsilon_1} \neq 0$. To determine the possible values of ϵ_1, y_{ϵ_1}, ϵ_2, y_{ϵ_2}, $\ldots$, it is necessary to consider the *Newton's diagram.*

Definition 4. (Newton's Diagram and Polygon). Given a pseudo-polynomial of the form (5) with coefficients given by (6), plot k versus ρ_k for $k = 0,1,\ldots,n$ (if $a_k(\cdot) \equiv 0$, the corresponding point is disregarded). Denote each of these points by $\pi_k = (k, \rho_k)$ and let

$$\Pi = \{\pi_k \,:\, a_k(\cdot) \neq 0\},$$

be the set of all plotted points. Then, the set Π will be called the *Newton diagram*, and the *Newton polygon* associated with $f(x, y)$ will be given by the lower boundary of the convex hull of the set Π.

For a given $f(x, y)$, Fig.1 simply illustrates Definition 4. Thus, the leading term of the expansion (7) of the solutions

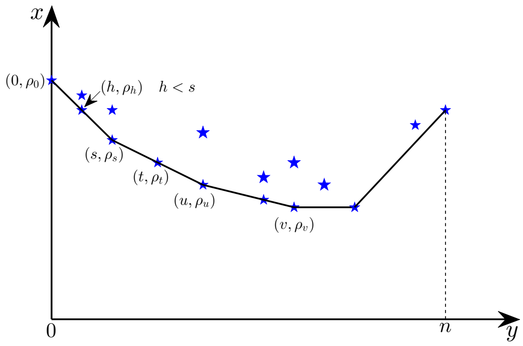


Fig. 1. The Newton Diagram for $f(x, y)$.

have exponents given by $\epsilon = \gamma$ where γ is the slope between two points of the Newton polygon. The coefficients are given by the non-zero solutions of the polynomial equation

$$\mathcal{P}(y_\epsilon) := \sum_i a_{0,i} y_\epsilon^i = 0, \tag{8}$$

where the sum runs over the terms satisfying $\rho_k + \gamma k = \nu$ with constant $\nu \in \mathbb{Q}$. For equations $f(x, y) = 0$, the *Newton Diagram Method* can be formalized by the following theorem (see Wall (2004)).

Theorem 5. (Puiseux Theorem). The equation $f(x, y) = 0$, with f given in formal power series such that $f(0, 0) = 0$, posses at least one solution in power series of the form:

$$x = t^q, \quad y = \sum_{i=1}^{\infty} c_i t^i, \quad q \in \mathbb{N}.$$

The procedure described above can be generalized to more than one parameter, to this end, we consider the following definitions.

2.4 Generalized Puiseux Series and Cones

When we deal with singularities of greater dimension, we must use a ring of multi-variable fractional power series. In McDonald (1995) the author defines fractional power series ring that contains the solutions of algebraic hypersurfaces. This is done through formal power series defined in a geometric way, by taking infinite power series

$$\sum_{i=1}^{\infty} c_{\boldsymbol{a_i}} \boldsymbol{x}^{\boldsymbol{a}_i/d}, \quad \text{where } \boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1} \cdots x_n^{a_n},$$

where the exponents $\boldsymbol{a}$ are taken from a fixed convex cone with a structure related to its Newton polytopes, Ziegler (2012). We will use fractional iterated power series of several variables as *Generalized Puiseux Power Series* (see Soto and Vicente (2011); Neumann (1949)), denoted by $K_{\boldsymbol{x},d}$. This series can be constructed by induction, taking as a bases the univariate case $K_{\boldsymbol{x}_1,d}$ and then, proceed with the field of power series in $x_1^{1/d}$ with power series coefficients in $x_2^{1/d} \cdots x_n^{1/d}$ such that

$$K_{\boldsymbol{x},d} = \mathbb{C}\left(\left(x_1^{1/d}\right)\right) \cdots \left(\left(x_n^{1/d}\right)\right).$$

2.5 Motivating Examples

Even though we can reduce the analysis of a given entire function f to the study of an algebraic function W, in this section we aim to point out some difficulties that arise in regarding multiparameter functions. In order to illustrate such arguments, let us consider the following motivating example.

Example 6. Consider the following polynomial

$$P(z, \epsilon_1, \epsilon_2) = z^2 + 3\epsilon_1 z + 2\left(\epsilon_1^2 + 2\epsilon_2^2\right), \tag{9}$$

where ϵ_1 and ϵ_2 are considered as perturbation parameters. It is clear to see, that for $\epsilon_1 = \epsilon_2 = 0$, $z = 0$ is a root of multiplicity two.

In this case, the solutions $z_{1,2}(\boldsymbol{\epsilon})$ are not analytic at $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2) = (0, 0) = \mathbf{0}$. Furthermore, $z_{1,2}(\boldsymbol{\epsilon})$ does not have a *unique* representation as a power series which is convergent in some punctured neighborhood of the origin. In order to illustrate this assertion, let us consider the region $|\epsilon_1| < |\epsilon_2|$, in this region the solutions admit the following representation

$$z_{1,2}(\boldsymbol{\epsilon}) = -\frac{1}{2}(3\epsilon_1 \pm \boldsymbol{i}4\epsilon_2) + \frac{1}{16}\epsilon_1\left(\pm\boldsymbol{i}\frac{\epsilon_1}{\epsilon_2} \pm \frac{\boldsymbol{i}}{64}\left(\frac{\epsilon_1}{\epsilon_2}\right)^3 + \pm\frac{\boldsymbol{i}}{2048}\left(\frac{\epsilon_1}{\epsilon_2}\right)^5 + \mathcal{O}\left(\left(\frac{\epsilon_1}{\epsilon_2}\right)^5\right)\right).$$

Now, if instead of the previous region, we consider the region $|\epsilon_2| < |\epsilon_1|$, then for $k \in \{1, 2\}$ the solutions admit the following representation

$$z_k(\boldsymbol{\epsilon}) = -2^{k-1}\epsilon_1 + (-1)^k 4\epsilon_2\left(\frac{\epsilon_2}{\epsilon_1} + 4\left(\frac{\epsilon_2}{\epsilon_1}\right)^3 + 32\left(\frac{\epsilon_2}{\epsilon_1}\right)^5 + \mathcal{O}\left(\left(\frac{\epsilon_1}{\epsilon_2}\right)^5\right)\right).$$

The above arguments clearly have shown that some further considerations must be taken into account in the case of multiparameter functions. Next, as mentioned in previous sections, in the single parameter case, the Newton diagram is a powerful tool to analyze the asymptotic behavior for the solutions of pseudo-polynomials. However, in order to be able to apply such a procedure to the multiparameter case, some special situations must be taken into consideration. In order to motivate the above arguments, let us consider the following.

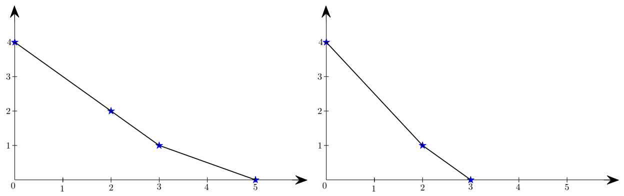
Example 7. Consider the polynomial

$$P(z, \boldsymbol{\epsilon}) := z^5 + \left(\epsilon_1\epsilon_2^3 + \epsilon_1^2\epsilon_2^2\right) z^3 + \left(\epsilon_1^2\epsilon_2^2 + \epsilon_1^3\epsilon_2\right) z^2 + \left(\epsilon_1^4\epsilon_2\right).$$

Clearly, $z = 0$ is a 5$-$multiple root at $\boldsymbol{\epsilon} = (0, 0)$. Now, let us form the Newton diagram with respect to ϵ_1, obtaining $\Pi = \{(0, 4), (2, 2), (3, 1), (5, 0)\}$, illustrated in Fig.2(a). The slope $\beta_0 = 1$ determines 3$-$solutions with respect to ϵ_1, and coefficients that are solutions of the polynomial

$$\mathcal{P}(\xi) = \epsilon_2 + \epsilon_2^2\xi^2 + \epsilon_2^3\xi^3 = 0.$$

In this case, it is clear that the solutions cannot be easily computed. In order to compute solutions by applying the



(a) Newton polygon with respect to ϵ_1. (b) Newton polygon with respect to y_2.

Fig. 2. Newton polygons for $P(z, \boldsymbol{\epsilon})$ in Example 7.

Newton procedure, we seek for a monic polynomial. Thus, with the aim of overcoming such difficulty, let us consider the change of variables (blowing-ups)

$$\zeta := \xi \quad \epsilon_1 = v_1 \epsilon_2 \quad \epsilon_2 = v_2,$$

and

$$v_1 = y_1 v_2 \quad v_2 = y_2.$$

In addition, these changes of variables will enable us to avoid horizontal segments in the subsequent steps of the process. The resulting polynomial $P'(y_1, y_2)$ posses the same Newton polygon, and the segment with $\beta_0 = 1$ has a monic polynomial

$$y_2^{-5}\mathcal{P}'(\xi, y_2) = y_2^4 + y_2\xi^2 + \xi^3 = 0.$$

Applying the Newton procedure, (see figure 2(b)) to the above equation, we derive the fractional power series solution of P':

$$\zeta_1(y_1, y_2) = -y_2 y_1 + o\left(y_1 y_2\right),$$

$$\zeta_{2,3}(y_1, y_2) = \pm \boldsymbol{i} y_2^{2/3} y_1 + o\left(y_1 y_2^{1/3}\right).$$

2.6 Problem Formulation

The present work is focused on computing the first approximation of the solution of quasi-polynomials around multiple imaginary roots. In this vein, we will focus in the following problems:

(i) compute an approximation of the associated Weierstrass polynomial;
(i) extend the Newton diagram procedure to Weierstrass polynomial of several variables;
(iii) obtain conditions that allow obtaining Puiseux series solutions

$$s(\tau_1, \tau_2) = c(\tau_2^{\beta})\tau_1^{\alpha} + o(\tau_1^{\alpha}\tau_2^{\beta}),$$

where α, $\beta \in \mathbb{Q}$.

3. MAIN RESULTS

3.1 Computation of Weierstrass Polynomial

In Mailybaev and Grigoryan (2001), the authors propose a method to compute the Weierstrass polynomial for an holomorphic function. This method is based on its partial derivatives and combinatorial factors related in a recursive way. For the case of holomorphic function, $f(z, \boldsymbol{x})$ of complex variables with $\boldsymbol{x} = (x_1, x_2)$ and $z = 0$ a m-multiple root at $(x_1, x_2) = (0, 0)$ the computation is given as follows. The coefficients w_i in (4) are analytic, $w_i(0, 0) = 0$ and can expressed in the form of Taylor series:

$$w_i(x_1, x_2) = \sum_{h_1+h_2=1}^{\infty} \frac{1}{h_1! h_2!} w_{i,\boldsymbol{h}} x_1^{h_1} x_2^{h_2},$$

where $\boldsymbol{h} = (h_1, h_2)$. The following notation is used:

$$w_{i,\boldsymbol{h}} = \left.\frac{\partial^{h_1+h_2} w_i}{\partial x_2^{h_1} \partial x_1^{h_2}}\right|_{(0,0)}, \quad F_{j,\boldsymbol{h}} = \left.\frac{\partial^{h_1+h_2} F_j}{\partial x_2^{h_1} \partial x_1^{h_2}}\right|_{(0,0)}.$$

Since the analytic function locally satisfy $f = Wb$, thus its partial derivatives satisfy the following recursive relations

$$w_{i,\boldsymbol{h}} = \sum_{j=0}^{i} \alpha_{ij} F_{j,\boldsymbol{h}}, \tag{10}$$

$$F_{j,\boldsymbol{h}} = f_{j,\boldsymbol{h}} - \sum_{k=0}^{j} \sum_{\boldsymbol{h}'+\boldsymbol{h}''=\boldsymbol{h}} c\left(j, k; \boldsymbol{h}', \boldsymbol{h}''\right) w_{k,\boldsymbol{h}'} b_{j-k,\boldsymbol{h}''},$$

with $\boldsymbol{h}' \neq \boldsymbol{0}$, $\boldsymbol{h}'' \neq \boldsymbol{0}$ and constant coefficients:

$$\alpha_{jj} = \frac{m!}{j! f_{m,\boldsymbol{0}}}, \quad \alpha_{ij} = -\frac{m!}{f_{m,\boldsymbol{0}}} \sum_{k=j}^{i-1} \frac{f_{m+i-k,\boldsymbol{0}} \alpha_{kj}}{(m+i-k)!},$$

$$c\left(j, k; \boldsymbol{h}_1, \boldsymbol{h}_2\right) = \frac{j!}{(j-k)!} \prod_{s=1}^{2} \frac{(h'_s + h''_s)!}{h'_s! h''_s!},$$

and for $\boldsymbol{h}' \neq 0$, $k' = k + m$, $b_{k,\boldsymbol{h}}$ is given by

$$\frac{k!}{(m+k)!}\left[f_{k',\boldsymbol{h}} - \sum_{j=0}^{m-1} \sum_{\boldsymbol{h}'+\boldsymbol{h}''=\boldsymbol{h}} c\left(k', j; \boldsymbol{h}', \boldsymbol{h}''\right) w_{j,\boldsymbol{h}'} b_{k'-j,\boldsymbol{h}''}\right].$$

Since we are only interested in the leading terms of w_i, namely a first approximation of the Weierstrass polynomial, we adopt the following notation.

Definition 8. Let the natural numbers $n_i^{(j)}$, for $i \in \{0, 1, \cdots, m-1\}$ and $j = 1, 2$, denote the first non zero partial derivative in (z, x_1, x_2) of f, such that the following conditions hold

$$f(0,0,0) = \frac{\partial^i f}{\partial z^i} = \cdots = \frac{\partial^{i+n_i^{(j)}-1} f}{\partial z^i \partial \tau_j^{n_i^{(j)}-1}} = 0, \quad \frac{\partial^{i+n_i^{(j)}} f}{\partial z^i \partial \tau_j^{n_i^{(j)}}} \neq 0,$$

with derivatives evaluated at $(0, \boldsymbol{0})$. For $n_i^{(j)} = \infty$ we have derivatives

$$\frac{\partial^i f}{\partial z^i} = \cdots = \frac{\partial^{i+n'_i-1} f}{\partial z^i \partial \tau_2^{n'_i-1}} = 0, \quad \frac{\partial^{i+n'_i} f}{\partial z^i \partial \tau_2^{n'_i}} \neq 0,$$

evaluated at $(z, \boldsymbol{x}) = (0, 0, 1)$.

Leading terms of coefficients w_i can be easy found up to the $n_i^{(j)}$ and n'_i derivatives, as a first observation we give the following result.

Proposition 9. Suppose that the Weierstrass polynomial has first non-zero partial derivative, such that

$$n_i^{(j)} > n_{i+1}^{(j)}, \quad 0 \leq i < m \text{ and } j = 1, 2.$$

Then, the leading terms of $w_i(\boldsymbol{x})$ are given by

$$w_i(x_1, x_2) = \alpha_{ii} f_{i,(n_i^{(1)},0)} x_1^{n_i^{(1)}} + \alpha_{ii} f_{i,(0,n_i^{(2)})} x_2^{n_i^{(2)}} + \cdots.$$

If $n_i^{(j)} = \infty$ we get

$$w_i(x_1, x_2) = \alpha_{ii} f_{i,(n'_i,\eta)} x_1^{n'_i} x_2^{\eta} + \cdots.$$

Remark 10. There may be a case in which

$$f_{i,(h_1,h_2)}\big|_{(0,0,0)} = 0 \quad \forall\, h_1, h_2 \in \mathbb{N}.$$

Since w_i are analytic functions, this is equivalent to $w_i(\boldsymbol{x}) \equiv 0$ for $0 \leq i \leq \kappa - 1$. Thus, according to Theorem 1 f has the following local structure:

$$z^{\kappa}\left[z^{m-\kappa} + w_{m-\kappa}(\boldsymbol{x}) z^{m-\kappa-1} + \cdots + w_{\kappa}(\boldsymbol{x})\right] b(z, \boldsymbol{x}).$$

3.2 The Newton Diagram Method for Two Parameters

Consider the monic pseudo-polynomial $f(z,\boldsymbol{x})$ of the form

$$z^m + a_{m-1}(x_1,x_2)z^{m-1} + \cdots + a_0(x_1,x_2), \tag{11}$$

with $a_i(\boldsymbol{x}) \in \mathbb{C}[[\boldsymbol{x}]]$, such that $f(0,\mathbf{0}) = 0$. The equation $f = 0$ can be solved by applying the Newton diagram method, this is done taking into account just one variable, say x_1, and proceeding iteratively. We take the point π_k as the order of a_k in x_1, taking x_2 as an element of $\mathbb{C}((x_2))$. For such a purpose, the following definition will be useful

$$\rho_k := \mathrm{ord}_{x_1}(a_k(x_1,x_2)) = \mathrm{ord}(a_k(x_1,1)). \tag{12}$$

Then, the *Newton Polygon* of $f(z,\boldsymbol{x})$, with respect to x_1, is defined by the lower boundary of the convex hull of the points $(k,\rho_k) \in \Pi$ (see, Definition 4). In order to apply the the Newton diagram procedure, according to Section 2.4, the solution z will take the following structure

$$z(x_1,x_2) = \sum_i c_i(x_2)x_1^{i/d},$$

where the coefficient $c_i(x_2)$, is in general, given by an univariate Puiseux series in x_2.

First Step into the Newton Procedure Let us suppose that we have determined the Newton diagram of the Weierstrass polynomial (11) of f. Since we are dealing with monic polynomials, the Newton polygon has a finite number of segments, each one with a corresponding set of points $\Pi^{(\ell)}$ and rational numbers $\beta_\ell \geq 0$ satisfying

$$\beta_0 > \beta_1 > \cdots > \beta_r.$$

Therefore, the segments are presented in two possible ways. The first one corresponds to a Newton polygon with a horizontal segment with $\beta_i = 0$, and the second one where $\beta_j > 0$ (for $i \neq j$). In this vein, for $0 \leq \ell < m$, the Newton Diagram Π is given as the set $\Pi = \Pi' \cup \Pi''$:

$$\{(0,\rho_0),\ldots,(\ell,0)\} \bigcup \{(\ell,0),\ldots,(k,\rho_k),\ldots,(m,0)\}.$$

Lets take at the first step of the process a horizontal segment with slope $\beta_r = 0$. We have the next two propositions.

Proposition 11. Let $f(z,\boldsymbol{x})$ be a pseudo-polynomial with the same structure as (11). Suppose that at least one coefficient $a_i(\boldsymbol{x})$ posses order $\rho_i = 0$. Then, the equation $\mathcal{P}(\xi,x_2) = 0$ (8) of the corresponding horizontal segment has solutions $c_k(x_2^{1/d})$ in the form of Puiseux series.

Now, at the first step of the process, the case with negative slope is considered. Hence, applying to f the change of variables $z = \zeta$, $x_1 = y_1^{a_1}$ and $x_2 = y_2^{a_2}$ we get $\tilde{f}(\zeta,y_1,y_2)$.

Proposition 12. Assume that f has the same structure as (11) and assume that the first Newton diagram posses a segment with negative slope. Then, there exist a change of variables $(z,x_1,x_2) \mapsto (\zeta,y_1,y_2)$ such that the polynomial $\mathcal{P}(\xi,y_2)$ has Puiseux series solutions $c_k(y_2^{1/d})$.

The iterative process continues by solving $\mathcal{P} = 0$, using the usual Newton diagram procedure.

3.3 Newton Polygon Algorithm

Let us consider the points $\pi_\ell = (\ell,\rho_\ell^{(1)}) \in \Pi$ to get the Newton polygon, obtaining a finite number of segments with slopes $-\beta_r$. Now, based on the Newton procedure introduced in Section 2.3 we propose Algorithm 1. In the algorithm, κ is defined according to Remark 10.

Algorithm 1 Auxiliary Puiseux Series Expansion

Let $f(s,\boldsymbol{\tau})$ have a root $s^* = \boldsymbol{i}\omega^*$ of multiplicity m at $\boldsymbol{\tau} = (\tau_1^*,\tau_1^*)$. Consider the initial values as $r := 0$, $i_{-1} := \kappa$, $\ell_{-1} := n_\kappa$.

1) Set $\mathcal{E}_r := \left\{\frac{\ell-\ell_{r-1}}{i_{r-1}-i} : (i,\ell) \in \Pi,\ \text{and}\ i > i_{r-1}\right\}$.

2) Let $\beta_r := \max \mathcal{E}_j$ and $\Pi^{(r)} := \{(i_{r-1},\ell_{r-1})\} \cup \left\{(i,\ell) \in \Pi : \beta_r \equiv \frac{\ell-\ell_{r-1}}{i_{r-1}-i}\right\}$.

3) Set $(i_r,\ell_r) \in \Pi^{(r)}$ such that $i_r \geq i,\ \forall (i,\ell) \in \Pi^{(r)}$.

4) Set $m_r := i_r - i_{r-1}$ and $r = r+1$.

5) If $i_{r-1} < m$ go to step 1. Otherwise the algorithm ends.

3.4 Puiseux Series for Quasi-Polynomials with two delays

Since any critical solution (s^*,τ_1^*,τ_2^*) can always be translated to the origin by appropriate shifts $s \mapsto s - s^*$, $\tau_1 \mapsto \tau_1 - \tau_1^*$, $\tau_2 \mapsto \tau_2 - \tau_2^*$, hereinafter we will assume that $(s^*,\tau_1^*,\tau_2^*) = (0,0,0)$.

Proposition 13. Consider the following quasi-polynomial

$$f(s,\tau_1,\tau_2) = p_0(s) + p_1(s)e^{-s\tau_1} + p_1(s)e^{-s\tau_2}, \tag{13}$$

with $s = 0$ a m-multiple root at $\boldsymbol{\tau} = (0,0)$ and local representation $f(s,\boldsymbol{\tau}) = W(\boldsymbol{\tau})b(s,\boldsymbol{\tau})$. If $n_i^{(j)} = 0$ for $i = 0,1,\ldots,k$ then, the $k+1$ coefficients of the Weierstrass polynomial W satisfy

$$w_{m-i}(\boldsymbol{\tau}) \equiv 0, \qquad i \in \{0,1,\ldots,k\}.$$

Proposition 14. Let quasi-polynomial $f(s,\boldsymbol{\tau})$ have a m-multiple roots $s = 0$ at $\boldsymbol{\tau} = (0,0)$, with associated Weierstrass polynomial W. Assume that

$$\mathcal{R}\left(W,\frac{\partial W}{\partial s}\right) = \tau_1^{a_1}\tau_2^{a_2}\mathcal{U}(\tau_1,\tau_2) \quad \text{such that } \mathcal{U}(0,0) \neq 0,$$

where $(a_1,a_2) \in \mathbb{Z}^2_{\geq 0} \setminus \{\mathbf{0}\}$, $\mathcal{U} \in \mathbb{C}\{\tau_1,\tau_2\}$. Then, $f = 0$ posses m solutions given by a generalized Puiseux series.

Finally, the following result gives some conditions to have a regular Newton diagram.

Proposition 15. Let $W(s,\tau_1,\tau_2)$ be the Weierstrass polynomial of a given quasi-polynomial $f(\tau_1,\tau_2)$. Assume that for a given $\ell-$segment of the Newton diagram, be $-\beta_\ell < 0$ its slope with corresponding points $\Pi^{(\ell)} = \{(k_1,\rho_{k_1}),(k_2,\rho_{k_2}),\ldots,(k_s,\rho_{k_s})\}$. Then, the equation $\mathcal{P}$ can be solved without any change of variables if the leading terms of the coefficients w_{k_i} satisfy

$$w_{k_i,(\rho_{k_i},\eta_{k_i})} \neq 0 \text{ whenever } \eta_{k_i} > \eta_{k_s}, \quad i < s.$$

.

4. NUMERICAL EXAMPLE

Example 16. Consider the following quasi-polynomial

$$f(s,\boldsymbol{\tau}) = \left(s^2 - 2s + 1\right) - 2e^{-s\tau_1} + 2\pi s e^{-s\tau_2} + e^{-2s\tau_2},$$

with $\boldsymbol{\tau} = (1,\pi)$, we have a triple root at $s = 0$. In order to apply the proposed results, let us consider $\tilde{f}(s,\boldsymbol{\tau}) := f(s,\tau_1+1,\tau_2+\pi)$. Now, by the Weierstrass Preparation Theorem, we know that the local behavior around the solution $\mathbf{0}$ of $\tilde{f}$, are captured by the solutions of $s^3 + w_2(\tau)s^2 + w_1(\tau)s + w_0(\tau)$. Since, the first non-zero partial derivatives are such that $n_0^{(1)} = n_0^{(2)} = \infty$, for $h_1 = (1,0), h_2 = (0,1)$, we have according to Remark 10 the natural numbers $n_1^{(j)}$

$$f_{1,h_j} = (-1)^{j+1}2 \Rightarrow n_1^{(j)} = 1, \quad j = 1,2.$$

Similarly, for $n_2^{(j)}$ we have

$$f_{2,h_1} = -4, \quad f_{2,h_2} = 4\pi \quad \Rightarrow n_2^{(j)} = 1.$$

Thus, applying Proposition 9 we obtain

$$w_1(\boldsymbol{\tau}) = \frac{6}{1-\pi^3}\tau_1 - \frac{6}{1-\pi^3}\tau_2 + \cdots$$

$$w_2(\boldsymbol{\tau}) = \frac{-3\left(3+4\pi^3(\pi-1)\right)}{2\left(\pi^3-1\right)^2}\tau_1 + \frac{3\left(4\pi-1\right)}{2\left(\pi^3-1\right)}\tau_2 + \cdots.$$

Now, by taking the set of points $\Pi'' = \{(1,0),(2,0),(3,0)\}$ as input of Algorithm 1, we get a horizontal segment with $\beta_0 = 0$, deriving the polynomial $\mathcal{P}(\xi,\tau_2)$. Hence, applying the Newton method to

$$\mathcal{P}(\xi,\tau_2) = w_1(0,\tau_2) + w_2(0,\tau_2)\xi + \xi^2,$$

we derive the leading terms of the solutions. Then, following the Newton procedure we get the points $\Pi' = \{(0,1)(1,1),(2,0)\}$. Thus, its Newton polygon has slope $\gamma = 1/2$ and associated polynomial

$$\mathcal{P}(\xi) := -\frac{6}{1-\pi^3} + \xi^2 = 0.$$

Therefore, we conclude that the first terms of solutions of f around $\mathbf{0}$ are given by

$$s_1(\tau_1,\tau_2) = 0,$$

$$s_{2,3}(\tau_1,\tau_2) = \pm\sqrt{\frac{6}{1-\pi^3}}\tau_2^{1/2} + o\left(\tau_1^{1/2}\tau_2^{1/2}\right).$$

5. CONCLUSION

In this paper, we have considered some issues concerning the asymptotic behavior of multiple critical roots for quasi-polynomials with two delays. The presented method is based on the Weierstrass Preparation Theorem which allows to deeply analyze the local behavior of a given solution. The proposed approach, using an extended Newton diagram method, can be effectively applied to find power series solutions in the form of generalized Puiseux series. Finally, we gave some conditions in which the solutions possess a regular behavior in the form of Puiseux series.

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