A new stability criterion for neutral-type systems with one delay ⋆

Marco A. Gomez * **Alexey V. Egorov** ** **Sabine Mondié** *

* *Departamento de Control Automático, Cinvestav, IPN, Mexico City, (e-mail: mgomez, smondie@ ctrl.cinvestav.mx).*
** *St. Petersburg State University, St. Petersburg, Russia (e-mail: alexey.egorov@spbu.ru).*

Abstract: A new necessary and sufficient exponential stability condition for time-delay systems of neutral-type is provided. Its main distinctive is that, analogously to the delay free case, it is given in terms of the delay Lyapunov matrix and requires a finite number of mathematical operations in order to be checked. The new stability criterion is illustrated by one example.

Keywords: Neutral-type delay systems; Stability criterion; delay Lyapunov matrix; Lyapunov-Krasvoskii functionals.

1. INTRODUCTION

Stability properties of neutral-type systems with delays have been studied in the last decades due to their effectiveness in describing a wide variety of physical phenomena (see, for example, Kolmanovskii and Myshkis (1999)). A powerful tool to analyze this class of systems has been the Lyapunov-Krasovskii functionals (see Niculescu (2001) and Fridman (2014)).

The first contribution addressing the computation of Lyapunov-Krasovskii functionals for neutral-type systems with prescribed derivative was presented in Castelan and Infante (1979). Based on the same ideas, the so-called functionals of complete type were introduced later for the one delay case in Rodriguez et al. (2004), Kharitonov (2005), and for the scalar case with multiple delays in Velázquez-Velázquez and Kharitonov (2009). One characteristic of the functionals of complete type introduced there is that they are determined by the delay Lyapunov matrix, which is solution of three equations, known as dynamic, symmetry and algebraic properties, and plays the role of the analogous of the Lyapunov matrix in the delay free case. This approach has been successfully applied to the estimation of exponential decay rate by Kharitonov (2005), computation of critical frequencies and parameters in Ochoa et al. (2013) and robust stability analysis by Alexandrova (2018), just to name a few. A comprehensive study of functionals of complete type and the delay Lyapunov matrix is available in the book by Kharitonov (2013).

The extension of the well-known stability criterion for delay free systems, which is given in terms of a matrix V, solution of the Lyapunov equation $A^T V + VA = -W$, has been subject of study in the last years. Recently, in the Lyapunov-Krasovskii functionals of complete type framework, necessary stability conditions for neutral-type systems have been presented in Gomez et al. (2017c) for one delay and in Gomez et al. (2017b) for multiple commensurate delays. The main characteristic of them is that, as in the delay free case, they depend uniquely on the delay Lyapunov matrix. A natural query is whether it is possible to obtain also sufficient stability conditions given in terms of the delay Lyapunov matrix or not.

A positive answer has been given for the retarded type case in Egorov et al. (2017), where an *infinite* stability criterion (i.e., it is such that an infinite number of mathematical operations is needed in order to check the stability of the system) is presented. Nevertheless, its infinite nature makes it only theoretical. In Egorov (2016), this drawback is eliminated by considering initial functions from a compact set and approximating them by a particular class of functions that depends on the fundamental matrix of the system, and a *finite* stability criterion (i.e. that requires only a finite number of mathematical operations) is obtained. However, the problem remains open for the neutral case.

Inspired by the ideas introduced in Egorov (2016) and Alexandrova and Zhabko (2016), we present a *finite* stability criterion for neutral-type systems with a single delay, which is given in terms of the positivity of a matrix constructed with the delay Lyapunov and fundamental matrices. It is worth mentioning that up to our knowledge, no stability criterion for neutral-type systems within the time-domain approach has been reported in the literature until now.

The rest of the paper is organized as follows. In the next section, we introduce some basic facts on the system. The fundamental framework of the delay Lyapunov matrix and Lyapunov-Krasovskii functionals is presented in Section 3. The necessary stability conditions depending on the delay Lyapunov matrix are recalled in Section 4. In Section 5, we provide some auxiliary results that allow us to obtain the main contribution of the paper: a new stability criterion for neutral-type systems with a single delay, which is presented in Section 6. We illustrate the result with one

⋆ This work was partially supported by Project CONACYT 180725.

example and conclude with some final remarks in Section 7 and Section 8, respectively.

Notation: The space of $\mathbb{R}^n$-valued piecewise continuous and continuously differentiable functions on $[-h,0]$ is denoted by $PC([-h,0],\mathbb{R}^n)$ and $C^{(1)}([-h,0],\mathbb{R}^n)$, respectively. For vectors and matrices, we use the Euclidian norm, denoted by $\|\cdot\|$, and for functions, we use the norm

$$||\varphi||_h = \max\left\{||\varphi(0) - D\varphi(-h)||, \sup_{\theta\in[-h,0]} ||\varphi(\theta)||\right\}.$$

The evaluation of the function G at point t on the right-hand (left-hand) side is denoted by $G(t+0)$ $(G(t-0))$. The notation $Q > 0$ means that matrix Q is positive definite. The maximum (minimum) eigenvalue of a matrix Q is represented by $\lambda_{\max}(Q)$ $(\lambda_{\min}(Q))$. The function that maps x to the least integer greater or equal to x is denoted by $\lceil x \rceil$. The square block matrix with $i-th$ row and $j-th$ column element Q_{ij} is represented by $[Q_{ij}]_{i,j=1}^r$.

2. BASIC FACTS ON THE SYSTEM

Consider the neutral-type delay system

$$\frac{d}{dt}\left(x(t) - Dx(t-h)\right) = A_0 x(t) + A_1 x(t-h), \quad (1)$$

where D, A_0 and A_1 belong to $\mathbb{R}^{n\times n}$ and $h > 0$ is the delay. The solution $x(\cdot,\varphi)$ of system (1) is piecewise continuous and satisfies the following:

(1) $x(\theta,\varphi) = \varphi(\theta)$, $\theta \in [-h,0]$,
(2) sewing condition: the function $x(t,\varphi) - Dx(t-h,\varphi)$ is continuous with respect to t (right continuous at $t=0$),
(3) $x(\cdot,\varphi)$ satisfies system (1) almost everywhere.

The initial function belongs to the space $PC([-h,0],\mathbb{R}^n)$ and the restriction of the solution $x(t,\varphi)$ to the interval $[t-h,t]$ is denoted by

$$x_t(\varphi) : \theta \to x(t+\theta,\varphi),\ \theta \in [-h,0].$$

Definition 1. System (1) is exponentially stable if there exist constants $\eta > 0$ and $\sigma > 0$ such that

$$\|x(t,\varphi)\| \le \eta e^{-\sigma t}\|\varphi\|_h,\ t \ge 0.$$

The fundamental matrix of system (1) is reminded in the next definition.

Definition 2. (Bellman and Cooke (1963)) The matrix K is known as the fundamental matrix of system (1) and is solution of the equation

$$\frac{d}{dt}\left(K(t) - DK(t-h)\right) = A_0 K(t) + A_1 K(t-h),\ a.e.$$

with the initial condition $K(0) = I$ and $K(t) = 0$ for $t < 0$.

The value of the fundamental matrix at points $t = jh$, $j = 0,1,\ldots$, coincides with the right-hand side value, i. e. $K(jh) = K(jh+0)$. The sewing condition implies that

$$\Delta K(jh) = D^j,\ j = 0,1,2\ldots,$$

where $\Delta K(jh) = K(jh) - K(jh-0)$.

It follows from Definition 2 that for $t \in [0,h]$, the fundamental matrix is given by

$$K(t) = \begin{cases} e^{A_0 t}, & t \in [0,h), \\ e^{A_0 h} + D, & t = h. \end{cases}$$

A well-known assumption in the stability study of system (1) is the Schur stability of matrix D (see Hale and Verduyn-Lunel (1993), Fridman (2014)). An upper estimate of the norm of this matrix is required in the subsequent results and one way for computing it is given in the next lemma.

Lemma 1. (Kharitonov et al. (2006)) A Schur stable matrix D admits the following upper bound:

$$\|D^k\| \le d\rho^k,$$

with $\rho \in (0,1)$ and $d = \sqrt{\dfrac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}}$, where $Q \in \mathbb{R}^{n\times n}$ is a positive definite matrix solution of

$$D^T Q D - \rho^2 Q < 0.$$

3. DELAY LYAPUNOV MATRIX FRAMEWORK

The delay Lyapunov matrix, denoted by U, is introduced in the next definition.

Definition 3. (Kharitonov (2013)) The delay Lyapunov matrix $U(\tau)$, $\tau \in [-h,h]$, associated with a positive definite matrix W, is a continuous matrix which satisfies:

(1) Dynamic property: For $\tau \in (0,h)$,
$$U'(\tau) - U'(\tau-h)D = U(\tau)A_0 + U(\tau-h)A_1, \quad (2)$$
(2) symmetry property: For $\tau \in [-h,h]$,
$$U^T(\tau) = U(-\tau), \quad (3)$$
(3) algebraic property:
$$\Delta U'(0) - D^T \Delta U'(0) D = -W, \quad (4)$$
where $\Delta U'(0) = U'(+0) - U'(-0)$.

The next theorem provides a criterion for the existence and uniqueness of matrix U.

Theorem 2. (Kharitonov (2013)) System (1) admits a unique Lyapunov matrix associated with a symmetric matrix W if and only if the system satisfies the Lyapunov condition, i.e., if there exists $\varepsilon > 0$ such that the sum of any two roots s_1 and s_2 of the spectrum of system (1) satisfies $|s_1 + s_2| > \varepsilon$.

The functional v_0 that satisfies

$$\frac{d}{dt}v_0(x_t(\varphi)) = -x^T(t,\varphi)Wx(t,\varphi),$$

for $\varphi \in PC([-h,0],\mathbb{R}^n)$, is determined by the delay Lyapunov matrix associated with the matrix W as follows (Kharitonov (2013), Gomez et al. (2016)):

$$\begin{aligned} v_0(\varphi) &= (\varphi(0) - D\varphi(-h))^T U(0)(\varphi(0) - D\varphi(-h)) \\ &+ 2(\varphi(0) - D\varphi(-h))^T \int_{-h}^0 F_1(-h-\theta)\varphi(\theta)d\theta \\ &+ \int_{-h}^0\int_{-h}^0 \varphi^T(\theta_1)F_2(\theta_1-\theta_2)\varphi(\theta_2)d\theta_2 d\theta_1 \\ &- \int_{-h}^0 \varphi^T(\theta)D^T\Delta U'(0)D\varphi(\theta)d\theta, \end{aligned}$$

where

$$F_1(\tau) = \begin{cases} U(\tau)A_1 + U'(\tau)D, & \tau \in [-h,h]\backslash\Omega, \\ 0, & \tau \in \Omega, \end{cases}$$

$$F_2(\tau) = \begin{cases} A_1^T F_1(\tau) - D^T F_1'(\tau), & \tau \in [-h,h]\backslash\Omega, \\ 0, & \tau \in \Omega, \end{cases}$$

where $\Omega = \{-h, 0, h\}$. Based on v_0, the following key functional was introduced in Gomez et al. (2017c):

$$v_1(\varphi) = v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta) W \varphi(\theta) d\theta, \tag{5}$$

whose derivative is

$$\frac{d}{dt} v_1(x_t(\varphi)) = -x^T(t-h,\varphi) W x(t-h,\varphi). \tag{6}$$

We now introduce the bilinear functional

$$\begin{aligned} z(\varphi_1, \varphi_2) = & \\ = & \left(\varphi_1(0) - D\varphi_1(-h)\right)^T U(0) \left(\varphi_2(0) - D\varphi_2(-h)\right) \\ & + \left(\varphi_1(0) - D\varphi_1(-h)\right)^T \int_{-h}^{0} F_1(-h-\theta) \varphi_2(\theta) d\theta \\ & + \int_{-h}^{0} \varphi_1^T(\theta) F_1^T(-h-\theta) d\theta \left(\varphi_2(0) - D\varphi_2(-h)\right) \\ & + \int_{-h}^{0} \int_{-h}^{0} \varphi_1^T(\theta_1) F_2(\theta_1 - \theta_2) \varphi_2(\theta_2) d\theta_2 d\theta_1 \\ & - \int_{-h}^{0} \varphi_1^T(\theta) \Delta U'(0) \varphi_2(\theta) d\theta, \end{aligned}$$

where $\varphi_1, \varphi_2 \in PC([-h, 0], \mathbb{R}^n)$, which also plays a key role in Gomez et al. (2017c). It is closely related to v_1: $v_1(\varphi) = z(\varphi, \varphi)$. We provide an upper bound for the previously mentioned functionals.

Lemma 3. For any $\varphi_1, \varphi_2 \in PC([-h, 0], \mathbb{R}^n)$,

$$|v_1(\varphi)| \leq \beta_2 \|\varphi\|_h^2,$$
$$|z(\varphi_1, \varphi_2)| \leq \beta_2 \|\varphi_1\|_h \|\varphi_2\|_h,$$

where

$$\beta_2 = \|U(0)\| + 2hf_1 + h^2 f_2 + h\|\Delta U'(0)\|,$$

with

$$f_1 = \sup_{\tau \in (0,h)} \|F_1(\tau)\|, \; f_2 = \sup_{\tau \in (0,h)} \|F_2(\tau)\|.$$

In the next theorem, it is shown that the functional v_1 satisfies a quadratic lower bound.

Theorem 4. If system (1) is exponentially stable,

$$v_1(\varphi) \geq \beta^\star \left\|\varphi(0) - D\varphi(-h)\right\|^2, \; \varphi \in PC([-h, 0], \mathbb{R}^n),$$

where $\beta^\star = \dfrac{\beta}{2}$ and $\beta > 0$ is such that

$$P(\beta) = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} + \beta \begin{pmatrix} A_0^T + A_0 & -A_0^T D + A_1 \\ -D^T A_0 + A_1^T & -A_1^T D - D^T A_1 \end{pmatrix} \geq 0.$$

Proof. Consider the functional

$$\widetilde{v}_1(\varphi) = v_1(\varphi) - \frac{1}{2}\int_{-h}^{0} \varphi^T(\theta) W \varphi(\theta) d\theta - \frac{\beta}{2}\|\varphi(0) - D\varphi(-h)\|^2.$$

Differentiating this functional with respect to the time, we obtain

$$\frac{d}{dt}\widetilde{v}_1(x_t) = -\frac{1}{2}\hat{x}^T(t) P(\beta) \hat{x}(t),$$

where $\hat{x}(t) = \left(x^T(t) \; x^T(t-h)\right)^T$. As system (1) is exponentially stable, $x(t) \to 0$ as $t \to \infty$, and

$$\lim_{t \to \infty} \int_0^t \frac{d}{ds}\widetilde{v}_1(x_s) ds = -\widetilde{v}_1(\varphi),$$

hence,

$$\widetilde{v}_1(\varphi) = \lim_{t \to \infty} \int_0^t \frac{1}{2}\hat{x}^T(s) P(\beta) \hat{x}(s) ds \geq 0,$$

which implies that

$$v_1(\varphi) \geq \frac{\beta}{2}\|\varphi(0) - D\varphi(-h)\|^2.$$

Remark 5. Since the greater the number $\beta^\star$ is, the stronger the inequality, we take β as the first value for which the determinant of the matrix $P(\beta)$ is zero.

4. NECESSARY STABILITY CONDITIONS

Let us introduce the following function, which depends on the fundamental matrix of system (1):

$$\psi_r(\theta) = \sum_{i=1}^{r} K(\tau_i + \theta)\gamma_i, \; \theta \in [-h, 0], \tag{7}$$

where $\gamma_i \in \mathbb{R}^n$ and $\tau_i \in [0, h]$, $i = \overline{1, r}$. In Gomez et al. (2017c), by the introduction of new properties of the delay Lyapunov matrix U that connect it with the matrix K, the next equality is obtained:

$$v_1(\psi_r) = \gamma^T \left[U(\tau_j - \tau_i)\right]_{i,j=1}^{r} \gamma, \tag{8}$$

where $\gamma = \left(\gamma_1^T \dots \gamma_r^T\right)^T$ and $\tau_i \in [0, h]$, $i = \overline{1, r}$. This equality is used in order to get a family of necessary stability conditions in terms of the delay Lyapunov matrix U for system (1). We recall it in the next theorem, which can be directly deduced from Theorem 4 and equality (8).

Theorem 6. (Gomez et al. (2017c)) If system (1) is exponentially stable, then the following condition holds:

$$\left[U(\tau_j - \tau_i)\right]_{i,j=1}^{r} > 0,$$

where $\tau_i \in [0, h]$, $i = \overline{1, r}$ and $\tau_i \neq \tau_j$ if $i \neq j$.

Theorem 6 and function (7) are the starting point for obtaining the stability criterion presented in Section 6.

5. AUXILIARY RESULTS

We define the set of initial functions

$$\mathcal{S} = \{\varphi \in C^{(1)}([-h, 0], \mathbb{R}^n) : \|\varphi\|_h = \|\varphi(0)\| = 1, \; \|\varphi'\| \leq \mu M\}$$

with $M = \|A_0\| + \|A_1\|$ and $\mu = \dfrac{d}{1-\rho}$, where the numbers d and ρ are given by Lemma 1. Notice that, like in Egorov (2016), the set $\mathcal{S}$ is also compact.

We present some auxiliary results related to the set $\mathcal{S}$ that are crucial in the attainment of the stability criterion. We introduce first a stability condition in terms of functional (5) and then we show that any arbitrary function from the set $\mathcal{S}$ can be approximated by a function of the form (7).

5.1 Sufficient stability condition in terms of v_1

The next lemma is useful for proving Theorem 8 below.

Lemma 7. (Gomez et al. (2017a)) Let P and Q be real matrices. If $\det(P + iQ) = 0$, then there exist two vectors C_1 and C_2 such that

(1) $(P + iQ)(C_1 + iC_2) = 0$.

(2) $\|C_1\| = 1$.
(3) $\|C_2\| \leq 1$.
(4) $C_1^T C_2 = 0$.

The basic idea of the following result is borrowed from Alexandrova and Zhabko (2016).

Theorem 8. Assume that matrix D is Schur stable. System (1) is exponentially stable if the Lyapunov condition holds and there exists $\beta_1 > 0$ such that for any $\varphi \in \mathcal{S}$

$$v_1(\varphi) \geq \beta_1. \tag{9}$$

Proof. Assume by contradiction that system (1) is not exponentially stable but the Lyapunov condition and inequality (9) hold. It means that there exists an eigenvalue $\lambda = \alpha + i\beta$ with $\alpha > 0$, and two vectors C_1, $C_2 \in \mathbb{R}^n$ that satisfy conditions of Lemma 7 such that

$$x(t, \varphi) = e^{\alpha t}\phi(t),\ \phi(t) = \cos(\beta t)\, C_1 - \sin(\beta t)\, C_2, \tag{10}$$

is a solution of system (1) on $t \in (-\infty, \infty)$. The initial function corresponding to solution (10) is given by

$$\varphi(\theta) = x(\theta, \varphi),\ \theta \in [-h, 0].$$

Let us prove first that $\varphi \in \mathcal{S}$. By Lemma 7, notice that $\|\varphi(0)\| = 1$ and $\|\phi(t)\|^2 = \cos^2(\beta t)\|C_1\|^2 + \sin^2(\beta t)\|C_2\|^2 \leq 1$. The last inequality implies that $\max_{t\in\mathbb{R}} \|\phi(t)\| = 1$, hence

$$\|x(t)\| = e^{\alpha t}\,\|\phi(t)\| \leq 1,\ t \leq 0.$$

Now, since $x(t, \varphi)$ satisfies (1) for $t \in (-\infty, \infty)$, we have

$$\begin{aligned}&\|\dot{x}(t) - D\dot{x}(t-h)\| \leq \\ &\quad \leq \|A_0\|\,\|x(t)\| + \|A_1\|\,\|x(t-h)\| \leq M,\ t \leq 0.\end{aligned}$$

The previous expression means that there is a function ξ that satisfies $\|\xi(t)\| \leq M$ for $t \leq 0$ and

$$y(t) = Dy(t-h) + \xi(t), \tag{11}$$

where $y(t) = \dot{x}(t)$. Notice that

$$y(t) = \sum_{j=0}^{\infty} D^j \xi(t - jh)$$

satisfies (11), indeed, by substituting it into (11), we get

$$y(t) - Dy(t-h) = \sum_{j=0}^{\infty} D^j\xi(t-jh) - \sum_{j=1}^{\infty} D^j\xi(t-jh) = \xi(t).$$

As matrix D is Schur stable, the sum converges and, by Lemma 1, there are constants $\rho \in (0, 1)$ and $d \geq 1$ such that $\left\|D^j\right\| \leq d\rho^j$, hence

$$\|y(t)\| \leq \sum_{j=0}^{\infty} d\rho^j M = \mu M,\ t \leq 0.$$

From the previous inequality we arrive at

$$\|\varphi'(\theta)\| = \|y(\theta)\| \leq \mu M,\ \theta \in [-h, 0].$$

Now, from equality (6),

$$v_1(\varphi) = v_1(x_T) + \int_{-h}^{T-h} x^T(t,\varphi) W x(t,\varphi) dt,$$

where $T = 2\pi/\beta$ if $\beta \neq 0$, and $T = 1$ if $\beta = 0$. Since T is the period of the function $\phi(t)$, we have $x(T+\theta) = e^{\alpha T}\varphi(\theta)$ and

$$v_1(x_T(\varphi)) = e^{2\alpha T} v_1(\varphi),$$

which implies that,

$$\begin{aligned}v_1(\varphi) &= -\frac{1}{e^{2\alpha T} - 1}\int_{-h}^{T-h} x^T(t,\varphi) W x(t,\varphi) dt \leq \\ &\leq -\frac{\lambda_{\min}(W)}{e^{2\alpha T} - 1}\int_{-h}^{T-h} \|x(t)\|^2 dt.\end{aligned}$$

The previous inequality contradicts the assumption and ends the proof.

5.2 Approximation of functions from the set $\mathcal{S}$

Consider a function $\varphi \in \mathcal{S}$. We construct the function ψ_r given by (7) as in Egorov (2016):

(1) Set $\tau_i = (i-1)\delta_r$, where $\delta_r = \dfrac{h}{r-1}$ and $r \geq 2$.
(2) Choose vectors γ_i, $i = \overline{1, r}$, such that
$$\psi_r(-\tau_i) = \varphi(-\tau_i). \tag{12}$$

Constructing the function ψ_r in such a way enables us to approximate any function φ from the space $\mathcal{S}$ and provide an estimate of the error, denoted by $R_r = \varphi - \psi_r$.

Lemma 9. For every $\varphi \in \mathcal{S}$

$$\|R_r\|_h = \|\varphi - \psi_r\|_h \leq \varepsilon_r,$$

where

$$\varepsilon_r = \frac{(\mu M + L)e^{Lh}}{1/\delta_r + L},$$

with L such that $\|K'(t)\| \leq L$, $t \in (0, h)$.

Proof. By equation (12), $R_r(-\tau_i) = 0$, hence,

$$\begin{aligned}\|R_r\|_h &= \sup_{\theta\in[-h,0]} \|\varphi(\theta) - \psi_r(\theta)\| \\ &= \max_{i\in\{2,\ldots,r\}} \sup_{\theta\in(-\tau_i,-\tau_{i-1})} \|\varphi(\theta) - \psi_r(\theta)\|.\end{aligned}$$

As $\|\varphi'(\theta)\| \leq \mu M$, then

$$\|\varphi(\theta) - \varphi(-\tau_i)\| \leq \mu M(\theta + \tau_i),\ \theta \in (-\tau_i, -\tau_{i-1}). \tag{13}$$

Observe that $\|K'(t)\| \leq L$, for $t \in (0, h)$, implies that

$$\|K(t_1) - K(t_2)\| \leq L|t_1 - t_2|,\ t_1, t_2 \in (0, h). \tag{14}$$

Now, take a number $i \in \{2, \ldots, r\}$. From (12), (13) and (14), we obtain the next sequence of inequalities:

$$\begin{aligned}\|\varphi(\theta) - \psi_r(\theta)\| &= \|\varphi(\theta) - \varphi(-\tau_i) + \psi_r(-\tau_i) - \psi_r(\theta)\| \\ &\leq \|\varphi(\theta) - \varphi(-\tau_i)\| + \|\psi_r(-\tau_i) - \psi_r(\theta)\| \\ &\leq (\tau_i + \theta)\left(\mu M + L\sum_{k=i}^{r}\|\gamma_k\|\right),\ \theta \in (-\tau_i, -\tau_{i-1}).\end{aligned}$$

We look for an upper bound estimate of $\|\gamma_k\|$. By equation (12), we have

$$\|\gamma_r\| = \|\varphi(-\tau_r)\| \leq \|\varphi(0)\| = 1,$$

and for $i = \overline{2, r-1}$

$$\sum_{k=i}^{r} K(\tau_k - \tau_i)\gamma_k = \gamma_i + \sum_{k=i+1}^{r} K(\tau_k - \tau_i)\gamma_k = \varphi(-\tau_i).$$

In view of the previous equality, we obtain

$$\|\gamma_i\| = \left\|\varphi(-\tau_i) - \sum_{k=i+1}^{r} K(\tau_k - \tau_i)\gamma_k\right\| =$$

$$
\begin{aligned}
=&\|\varphi(-\tau_i)-\varphi(-\tau_{i+1})+\psi_r(-\tau_{i+1})\\
&-\sum_{k=i+1}^{r}K(\tau_k-\tau_i)\gamma_k\|\le\|\varphi(-\tau_i)-\varphi(-\tau_{i+1})\|\\
&+\sum_{k=i+1}^{r}\|K(\tau_k-\tau_{i+1})-K(\tau_k-\tau_i)\|\,\|\gamma_k\|\\
\le&\mu M\delta_r+L\delta_r\sum_{k=i+1}^{r}\|\gamma_k\|.
\end{aligned}
$$

Using the preceding inequality, one can prove by induction that

$$\|\gamma_i\|\le\delta_r\left(\mu M+L\right)\left(1+\delta_r L\right)^{r-i-1},\ i=\overline{2,r-1},$$

therefore,

$$
\begin{aligned}
&\mu M+L\sum_{k=2}^{r}\|\gamma_k\|\le\\
&\qquad\le\mu M+L\delta_r\left(\mu M+L\right)\sum_{k=2}^{r-1}\left(1+\delta_r\right)^{r-k-1}+L.
\end{aligned}
$$

Expanding the sum and rearranging terms in the right hand side, we arrive at

$$\mu M+L\sum_{k=2}^{r}\|\gamma_k\|\le\left(\mu M+L\right)\left(1+L\delta_r\right)^{r-2}.$$

Finally, since $(\tau_i+\theta)\le\delta_r$ for $\theta\in(-\tau_i,-\tau_{i-1})$, we have

$$
\begin{aligned}
\|R_r\|_h\le&\max_{i\in\{2,\dots,r\}}\sup_{\theta\in(-\tau_i,-\tau_{i-1})}\left(\mu M+L\sum_{k=i}^{r}\|\gamma_k\|\right)\delta_r\\
\le&\frac{\delta_r(\mu M+L)}{1+L\delta_r}\left(1+\frac{Lh}{r-1}\right)^{r-1}\le\frac{(\mu M+L)e^{Lh}}{1/\delta_r+L}.
\end{aligned}
$$

Remark 10. The estimate of the supremum norm of the error R_r is of the same form as the one obtained in Egorov (2016) for the retarded type case, except for the term μ, which indeed is related to the matrix D.

6. STABILITY CRITERION

We set τ_i as in Subsection 5.2 and consider the matrices with constant coefficients

$$\mathcal{K}_r=\left[U(\tau_j-\tau_i)\right]_{i,j=1}^{r}=\left[U\left(\frac{j-i}{r-1}h\right)\right]_{i,j=1}^{r},$$

and

$$
\mathcal{A}_r=
\begin{pmatrix}
K^T(\tau_1)K(\tau_1) & K^T(\tau_1)K(\tau_2) & \dots & K^T(\tau_1)K(\tau_r-0)\\
\star & K^T(\tau_2)K(\tau_2) & \dots & K^T(\tau_2)K(\tau_r-0)\\
\vdots & & \ddots & \vdots\\
\star & \star & \dots & K^T(\tau_r-0)K(\tau_r-0)
\end{pmatrix}.
$$

Henceforth, we assume that $\mathcal{K}_1=U(0)$. In the next theorem, we provide the main result of the paper: a new stability criterion for system (1) for the case in which $\|D\|<1$.

Theorem 11. Assume that matrix D satisfies $\|D\|<1$. System (1) is exponentially stable if and only if the Lyapunov condition and the following hold:

$$\mathcal{K}_r-\beta_1\mathcal{A}_r>0, \tag{15}$$

where

$$r=1+\left\lceil e^{Lh}h\left(\mu M+L\right)\left(\alpha+\sqrt{\alpha(\alpha+1)}\right)-Lh\right\rceil, \tag{16}$$

with $\alpha=\dfrac{\beta_2}{\beta_1(1-\|D\|)^2}$. Here, $\beta_1\in(0,\beta^\star)$, $\beta^\star$ is provided by Theorem 4 and β_2 is given by Lemma 3.

Proof. *Necessity:* Consider the function (7). By the initial conditions of the fundamental matrix and from the fact that $\tau_r=h$, the following chain of equalities holds for every $\gamma_i\in\mathbb{R}^n$, $i=\overline{1,r}$:

$$
\begin{aligned}
&\psi_r(0)-D\psi_r(-h)=\sum_{i=1}^{r}K(\tau_i)\gamma_i-D\sum_{i=1}^{r}K(\tau_i-h)\gamma_i=\\
&\sum_{i=1}^{r}K(\tau_i)\gamma_i-DK(\tau_r-h)\gamma_r=\sum_{i=1}^{r-1}K(\tau_i)\gamma_i+K(h-0)\gamma_r,
\end{aligned}
$$

which implies that

$$\|\psi_r(0)-D\psi_r(-h)\|^2=\gamma^T\mathcal{A}_r\gamma,$$

where $\gamma=\begin{pmatrix}\gamma_1^T & \dots & \gamma_r^T\end{pmatrix}^T$. By Theorem 4 and equality (8), we have, for every $\gamma\in\mathbb{R}^{nr}$ such that $\gamma^T\mathcal{A}_r\gamma>0$,

$$
\begin{aligned}
\gamma^T\mathcal{K}_r\gamma-\beta_1\gamma^T\mathcal{A}_r\gamma&=v_1(\psi_r)-\beta_1\|\psi_r(0)-D\psi_r(-h)\|^2\\
&>v_1(\psi_r)-\beta^\star\|\psi_r(0)-D\psi_r(-h)\|^2\ge0.
\end{aligned}
$$

For the case in which $\gamma^T\mathcal{A}_r\gamma=0$, $\gamma\ne0$, the inequality $\gamma^T\mathcal{K}_r\gamma-\beta_1\gamma^T\mathcal{A}_r\gamma>0$ remains true, since from Theorem 6, for every number r, $\mathcal{K}_r>0$.

Sufficiency: Consider a function $\varphi\in\mathcal{S}$ and $R_r=\varphi-\psi_r$ and observe that

$$
\begin{aligned}
v_1(\varphi)&=z(\psi_r+R_r,\psi_r+R_r)\\
&=z(\psi_r,\psi_r)+2z(\psi_r,R_r)+z(R_r,R_r)\\
&=v_1(\psi_r)+2z(\varphi,R_r)-v_1(R_r).
\end{aligned}
$$

By construction, $\psi_r(0)=\varphi(0)$, $\psi_r(-h)=\varphi(-h)$, and $\|\varphi(0)\|=1$, hence, from Lemma 3 and the fact that $\|D\|<1$, we get

$$
\begin{aligned}
&v_1(\varphi)\ge v_1(\psi_r)-\beta_1\|\psi_r(0)-D\psi_r(-h)\|^2\\
&+\beta_1\|\varphi(0)-D\varphi(-h)\|^2-2\beta_2\|R_r\|_h-\beta_2\|R_r\|_h^2\ge\\
&\ge v_1(\psi_r)-\beta_1\|\psi_r(0)-D\psi_r(-h)\|^2\\
&+\beta_1(1-\|D\|)^2-2\beta_2\|R_r\|_h-\beta_2\|R_r\|_h^2.
\end{aligned}
$$

For the number r given by (16), it follows from Lemma 9 that

$$\|R_r\|_h\le\frac{(\mu M+L)e^{Lh}h}{r-1+Lh}\le\frac{\bar\beta_1}{\beta_2+\sqrt{\beta_2(\beta_2+\bar\beta_1)}},$$

where $\bar\beta_1=(1-\|D\|)^2\beta_1$. Notice that, from the above inequality, we have

$$\bar\beta_1-2\beta_2\|R_r\|_h-\beta_2\|R_r\|_h^2\ge0.$$

Therefore,

$$
\begin{aligned}
v_1(\varphi)&\ge v_1(\psi_r)-\beta_1\|\psi_r(0)-D\psi_r(-h)\|^2\\
&=\gamma^T\mathcal{K}_r\gamma-\beta_1\gamma^T\mathcal{A}_r\gamma\ge\lambda_{\min}\left(\mathcal{K}_r-\beta_1\mathcal{A}_r\right)\|\gamma\|^2.
\end{aligned}
$$

As $1=\|\psi_r(0)\|^2=\gamma^T\left[K^T(\tau_i)K(\tau_j)\right]_{i,j=1}^{r}\gamma$,

$$1\le\lambda_{\max}\left(\left[K^T(\tau_i)K(\tau_j)\right]_{i,j=1}^{r}\right)\|\gamma\|^2,$$

which implies that there exists a number $\tilde\gamma>0$ such that $\|\gamma\|\ge\tilde\gamma$, and in turn that

$$v_1(\varphi)\ge\widetilde\beta,$$

with $\widetilde\beta=\lambda_{\min}\left(\mathcal{K}_r-\beta_1\mathcal{A}_r\right)\tilde\gamma^2>0$. As $\|D\|<1$ implies Schur stability of the matrix D, one can use Theorem 8 and the previous inequality to conclude that system (1) is exponentially stable.

Notice that the number r, which depends on the parameters of the system, determines the size of the matrix $\mathcal{K}_r - \beta_1\mathcal{A}_r$ and in turn the numerical complexity. Indeed, one can see from equation (16) that r increases (or decreases) as the delay does.

7. EXAMPLE

We analyze system (1) with delay $h = 1$ and matrices

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}, A_0 = \begin{pmatrix} -0.1 & 0 \\ 0 & p \end{pmatrix}, A_1 = \begin{pmatrix} -0.2 & 0.1 \\ 0.1 & 0 \end{pmatrix}, \quad (17)$$

for two different values of the parameter $p \in \mathbb{R}$. The parameters to be calculated in order to use Theorem 11 are shown next. For $p = -0.2$,

$$\beta^\star = 1.1432,\ \beta_2 = 5.7693,\ \ M = 0.4414,\ L = 0.2443,$$

and for $p = 0.1$,

$$\beta^\star = 1.4388,\ \beta_2 = 6.4521,\ M = 0.3414,\ L = 0.1105.$$

The number $\mu = 1.1292$ remains equal in both cases. Table 1 shows the computed number r and the minimum eigenvalue of $\mathcal{K}_r - \beta_1\mathcal{A}_r$. According to the obtained results, it follows from Theorem 11 that system (17) is exponentially stable for $p = -0.2$ and unstable for $p = 0.1$. This is corroborated by the spectral abscissa computed via the QPmR (Quasi-Polynomial Mapping Based Rootfinder) algorithm (Vyhlídal and Zítek (2009)).

Table 1. Stability test of system (17)

Parameter	Number r	$\lambda_{min}\,(\mathcal{K}_r - \beta_1\mathcal{A}_r)$	Spectral abscissa
$p = -0.2$	$r = 14$	0.0195	-0.0518
$p = 0.1$	$r = 8$	-48.7391	0.1968

8. CONCLUSIONS

A new exponential stability criterion for neutral type delay systems is presented. As in the delay free case, the necessary and sufficient stability condition requires checking the positive definiteness of a special matrix constructed in terms of the delay Lyapunov and fundamental matrices and whose dimension depends on the system parameters.

Future work includes the non-trivial extension to the multiple delay case and the generalization to the case in which the matrix D does not satisfy $\|D\| < 1$.

REFERENCES

Alexandrova, I.V. (2018). New robustness bounds for neutral type delay systems via functionals with prescribed derivative. *Applied Mathematics Letters*, 76, 34–39.

Alexandrova, I.V. and Zhabko, A.P. (2016). Synthesis of Razumikhin and Lyapunov-Krasovskii stability approaches for neutral type time delay systems. In *Proceedings of System Theory, Control and Computing (ICSTCC)*, 375–380.

Bellman, R.E. and Cooke, K.L. (1963). *Differential-difference equations.* Academic Press, New York.

Castelan, W.B. and Infante, E.F. (1979). A Lyapunov functional for a matrix neutral difference-differential equation with one delay. *J. of mathematical Analysis and Applications*, 71(1), 105–130.

Egorov, A.V., Cuvas, C., and Mondié, S. (2017). Necessary and sufficient stability conditions for linear systems with pointwise and distributed delays. *Automatica*, 80, 218–224.

Egorov, A.V. (2016). A finite necessary and sufficient stability condition for linear retarded type systems. In *Proceedings of the 55th IEEE Conference on Decision and Control*, 3155–3160. Las Vegas, USA.

Fridman, E. (2014). *Introduction to time-delay systems: Analysis and control.* Birkhäuser, Basel.

Gomez, M.A., Egorov, A., and Mondié, S. (2017a). A Lyapunov matrix based stability criterion for a class of time-delay systems. *Vestnik Sankt-Peterburgskogo Universiteta. Prikl. Mat., Inf., Prot. Upr.*, 13(4).

Gomez, M.A., Egorov, A., and Mondié, S. (2017b). Necessary stability conditions for neutral-type systems with multiple commensurate delays. *Int. J. of Control*, DOI: 10.1080/00207179.2017.1384574.

Gomez, M.A., Egorov, A.V., and Mondié, S. (2016). Obtention of the functional of complete type for neutral type delay systems via a new Cauchy formula. In *Proceedings of 13th IFAC Workshop on Time Delay Systems*, 118–123. Istanbul, Turkey.

Gomez, M.A., Egorov, A.V., and Mondié, S. (2017c). Necessary stability conditions for neutral type systems with a single delay. *IEEE Trans. on Automatic Control*, 62(9), 4691–4697.

Hale, J.K. and Verduyn-Lunel, S.M. (1993). *Introduction to Functional Differential Equations.* Springer Science + Business Media, New York.

Kharitonov, V.L. (2005). Lyapunov functionals and Lyapunov matrices for neutral type time delay systems: A single delay case. *Int. J. of Control*, 78(11), 783–800.

Kharitonov, V.L. (2013). *Time-Delay Systems: Lyapunov functionals and matrices.* Birkhäuser, Basel.

Kharitonov, V.L., Collado, J., and Mondié, S. (2006). Exponential estimates for neutral time delay systems with multiple delays. *Int. J. of Robust and Nonlinear Control*, 16, 71–84.

Kolmanovskii, V. and Myshkis, A. (1999). *Introduction to the theory and applications of functional differential equations.* Kluwer Academic Publishers, Dordrecht.

Niculescu, S.I. (2001). *Delay effects on stability: A robust control approach*, volume 269. Springer-Verlag London.

Ochoa, G., Kharitonov, V.L., and Mondié, S. (2013). Critical frequencies and parameters for linear delay systems: A Lyapunov matrix approach. *Systems & Control Letters*, 62(9), 781–790.

Rodriguez, S.A., Kharitonov, V.L., Dion, J.M., and Dugard, L. (2004). Robust stability of neutral systems: a Lyapunov-Krasovskii constructive approach. *Int. J. of Robust and Nonlinear Control*, 14(16), 1345–1358.

Velázquez-Velázquez, J.E. and Kharitonov, V.L. (2009). Lyapunov- Krasovskii functionals for scalar neutral type time delay equations. *Systems & Control Letters*, 58(1), 17–25.

Vyhlídal, T. and Zítek, P. (2009). Mapping based algorithm for large-scale computation of quasi-polynomial zeros. *IEEE Trans. on Automatic Control*, 54(1), 171–177.