Delays in Gambler's Ruin and Random Relays ⋆

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Abstract: We present here two examples of effects of delays on stochastic systems. The first model is an inclusion of delay in a classical problem of gambler's ruin, modeled by a random walk. The second model introduces delays in a lined collection of random walks which pass a baton from one end to the other. The effects of delays as well as possible applications of these models are discussed

Keywords: Delay, Random Walks, Stochastic System, Relay

1. INTRODUCTION

Studies of delays in dynamics, particularly in the context of feedback control systems, have found rather intricate and complex behaviors even for a simple first order differential equation.(6; 17). "Delay Differential Equations" are the main mathematical approaches and modeling tools for such systems. We can further introducing stochastic elements together with delays. For such systems, "Stochastic Delay Differential Equations" (4; 10) or "Delayed Random Walks"(12; 13) have been proposed and investigated. It has been pointed out that interplay between stochasticity and delay can give rise to a peculiar behavior such as "Delayed Stochastic Resonances"(11; 14; 19).

We present here two models incorporating stochasticity and delay. The first one is an extension of the classic problem of Gambler's Ruin(1) A gambler who has an initial asset takes on betting under certain probabilities of win and loss, until he is broken or reaches to a specific asset level. One of the simplest models can be described as a restricted random walk with two absorbing boundaries. The position of the walker indicates his asset and the walker can probabilistically gain or lose a unit of his asset at each time step, and moves accordingly until he reaches either boundary. This model has been studied extensively and various extensions are made (for examples (7; 8; 15; 16)).The main feature of our extended model, which we call "Delayed Gambler's Ruin"(DGR), is that it includes delays in the gain or loss of a unit in the gambler's asset. This reflects that, in reality, payments and/or incomes often do not take place immediately at the time of corresponding events, such as a purchase with a credit card. Our proposed model, thus, moves according to past results of gambling.

The second model also concerns with random walks. A collection of random walkers which are lined and move in one dimension passes a "baton" or a "message" from one to the next. The baton holder can pass it to the next one when they come in contact. We investigate behaviors during this relaying of the baton from the starting walker to the final one. Delays are introduced in each transfer of the baton, which we call "Delayed Random Relay"(DRR). In this model, each walker is required to hold the baton for a certain time period (delay) before it can pass to the next walker. We will investigate the effect of this inclusion of the delay on the total relaying time.

Through these models, we provide new examples which show interplays between stochasticity and delay.

2. DELAYED GAMBLER'S RUIN

We start with a brief description of the Gambler's Ruin. A gambler attends a gamble with the initial asset of x points. At each bet, he either wins or loses one point with probabilities p and $1-p$ respectively. He ends his betting either when his assets become zero ("broken") or reach to his intended asset level A. We now define some notations to analyze this problem.

- p: The probability of the gambler's winning a point. (We also set $q = 1-p$ as the probability of losing.)
- U_t: Gambler's asset after t-th betting.
- $P_A(x)$: Probability that a gambler, who has the initial asset of x points and the intended winning asset level of A, to become broken.
- $X_t = \pm 1$: The change of the asset at t-th betting.

By mapping this problem to restricted symmetric simple random walks with each step as X_t and with absorbing boundaries at 0 and A, following results are known.

$$P_A(x) = \begin{cases} \dfrac{A-x}{A} & (p = q = \dfrac{1}{2}) \\[2ex] \dfrac{(\frac{q}{p})^A - (\frac{q}{p})^x}{(\frac{q}{p})^A - 1} & (p \neq q) \end{cases} \tag{1}$$

We now extend the above basic model to investigate how this probability of ruin is affected by inclusion of delays. Two delays are introduced into the above Gambler's Ruin model:

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- τ_r: delay in receipt (winning) of a point.
- τ_p: delay in payment (losing) of a point.

At each bet, winning or losing of the point is deferred with the above delays. We denote the probability of ruin in the Delayed Gambler's Ruin as $P_A^{\tau_r,\tau_p}(x)$. We will focus on this probability in the following analysis. The stopping time, T_{τ_r,τ_p}, is also defined as the time duration of the gambler's betting (i.e., the time between the beginning to the end of his betting, either by broken or by reaching A)

It turns out the important parameter in analyzing this model is the difference between these two delays:

- $\theta = \tau_r - \tau_p$: the difference of delay in payment and receipt.

With the above setup, we start our analysis by considering different cases of θ. We assume that the initial asset is further away from the boundary than this difference between delays.

$$A - x > \mid \theta \mid \quad and \quad x > \mid \theta \mid$$

In the following, we present results of calculations and approximations. The details are found in (9).

The case $\theta = 0 \quad (\tau_r = \tau_p)$

We first consider the case of $\theta = 0$, which means two delays are the same, $\tau_r = \tau_p$.

It turns out that we can reduce the problem for the case of $\theta = 0$ to the original Gambler's Ruin, leading to the following ruin probability for $\tau_r = \tau_p$

$$P_A^{\tau_r,\tau_p}(x) = \begin{cases} \dfrac{A-x}{A} & (p = q = \dfrac{1}{2}) \\ \dfrac{(\frac{q}{p})^A - (\frac{q}{p})^x}{(\frac{q}{p})^A - 1} & (p \neq q) \end{cases} \tag{2}$$

The case $\theta > 0 \quad (\tau_r > \tau_p)$

Let us now consider the case when the delay in receipt is longer than that of payment, $\tau_r > \tau_p$ ($\theta > 0$).

In this case, the exact calculation is quite intricate, but we can employ an approximation by replacing some of the stochastic variables by its mean value in some steps of the calculation. This approximation leads to the ruin probability as

$$P_A^{\tau_r,\tau_p}(x) \approx \begin{cases} \dfrac{A - \{x - (1 + q(\theta - 1))\}}{A} & (p = q = \dfrac{1}{2}) \\ \dfrac{(\frac{q}{p})^A - (\frac{q}{p})^{(x-(1+q(\theta-1))}}{(\frac{q}{p})^A - 1} & (p \neq q) \end{cases} \tag{3}$$

We note that this is the same formula as no delay case, except that the value of the initial asset is decreased from x by the amount $1 + q(\theta - 1)$. This is a natural result considering that the receipt of points is more delayed than the payments. Hence, the gambler's asset tends to be lower at any time points, leading to a higher probability of ruin compared to the case of no delays or the same delays. This approximation accounts for these effects by shifting the initial assets to lower points.

The case $\theta < 0 \quad (\tau_r < \tau_p)$

We now consider the opposite case with $\tau_r < \tau_p$. By essentially the same arguments, the ruin probability for this case can be approximated as

$$P_A^{\tau_r,\tau_p}(x) \approx \begin{cases} \dfrac{A - \{x + p(-\theta - 1)\}}{A} & (p = q = \dfrac{1}{2}) \\ \dfrac{(\frac{q}{p})^A - (\frac{q}{p})^{(x+p(-\theta-1))}}{(\frac{q}{p})^A - 1} & (p \neq q) \end{cases} \tag{4}$$

We again notice that this is the same formula as no delay case, but the initial asset is now increased by the amount $p(-\theta - 1)$. As in the case for $\theta > 0$, we can see that this is an effective approximation to account for the decrease of the ruin probability using an increase of the initial asset points.

2.1 Comparison Against Computer Simulations

In this section, we will compare our approximate results for $P_A^{\tau_r,\tau_p}(x)$ with computer simulations.

We will fix the following parameters:

- $A = 100$
- $p = 9/19, \quad (q = 11/19)$

Also, we take 10,000 trials to obtain average values from computer simulations. We show only a representative examples, leaving other results to the separate paper (9).

The case $\theta > 0, \quad (\tau_r > \tau_p)$

For the simplicity, we set $\tau_p = 0$ and vary τ_r, and initial asset points x. A representative result is given in the following table and associated plots. The Column A, B are, respectively, the estimations from computer simulations, and from our approximation Eq. (3). Though data are limited due to constraints on computational times, for the ranges of $\tau_r (= \theta)$, the discrepancy is less than 2 point percentiles.

τ_r	A	B
1	0.6883	0.6862
2	0.7039	0.7031
3	0.7283	0.7192
4	0.7422	0.7343
5	0.7587	0.7486
6	0.7696	0.7622
7	0.7883	0.7750
8	0.8036	0.7872
9	0.8082	0.7986
10	0.8223	0.8095

Table 1. The case with $x = 90$

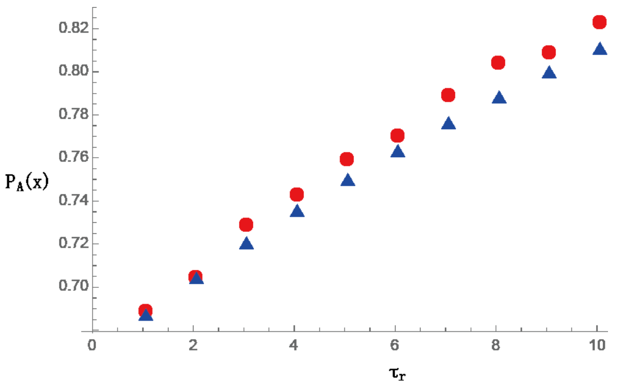

Fig. 1. Comparison of computer simulations A (red dots), and the analytical approximation B (blue triangles).

The case $\theta < 0, \quad (\tau_r < \tau_p)$

Again, for the simplicity, we set $\tau_r = 0$ and vary τ_p and initial asset points x. A representative result is given in the following table and associated plots. The Column A, B are, respectively, the estimations from computer simulations, and from our approximation Eq. (4).

τ_p	A	B
1	0.6461	0.6513
2	0.6360	0.6335
3	0.6241	0.6147
4	0.6092	0.5950
5	0.5889	0.5743
6	0.5774	0.5525
7	0.5571	0.5296
8	0.5342	0.5055
9	0.5094	0.4802

Table 2. The case with $x = 90$

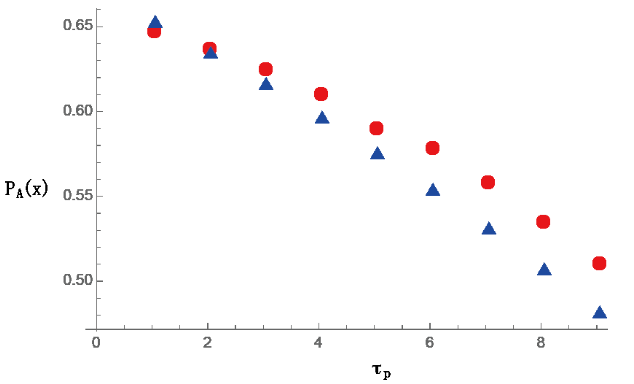

Fig. 2. Comparison of computer simulations A (red dots), and the analytical approximation B (blue triangles).

3. DELAYED RANDOM RELAYS

We now proceed to propose and study Delayed Random Relays(18). Let us start with a description of our model without delays. A schematic view of relaying by random walkers is given in Figure 3. We consider a one-dimensional line with a periodic boundary condition (a circle). On the line, there are N discrete sites on which random walkers hop. We place n simple symmetric random walkers on these sites. Each random walker, at each unit time, takes a unit step either to his right or left with the equal probabilities of 1/2.

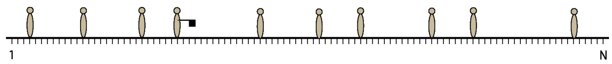

Fig. 3. We show the case with 10 walkers. The forth walker from the left is holding a baton.

A baton is relayed in one direction by this group of random walkers. When the baton holder moves to an adjacent position, i.e., in contact, to the next walker, the baton is passed on. We measure the time, T, for the baton to travel from the starting random walker to the last walker in the line.

We performed computer simulations of this model with the parameters of $N = 100$ and varying the number n of walkers in the relay. For each n, we repeated the simulations for 10000 trials and computed the average values of this total relaying time T which is plotted as a function of n (Figure 4). We observe a monotonic decrease of T, meaning that the more walkers we have, the faster the baton is relayed.

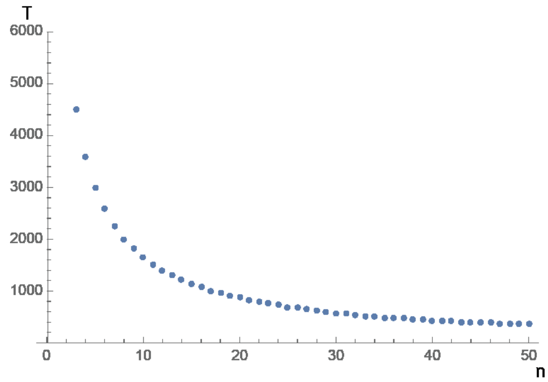

Fig. 4. Total relaying time T as a function of the number of walkers n.

Now we extend this basic model to include a delay in relaying of the baton. When the baton is received from the previous walker, it is not immediately transferable to the next walker. There is a time interval, which we call delay d, for the baton to become transferable. One can view this as a requirement for each walker to hold the baton for at least a certain period of time, or as a maturing time for the baton becoming ready to be passed on.

Hence, even though a baton holding random walker comes in contact with the next walker, he may not be able to pass the baton. Only after the other condition that the delay time has elapsed is also satisfied, the baton is passed on to the next walker.

With this extended model, we also measured the total relaying time T as a function of the number of walkers n. A representative result is shown in Figure 5. A notable feature is that there exists an optimal number of walkers for the fastest relaying of the baton.

The existence of the optimal number of walkers in the relay can be qualitatively understood as follows. With a large number of walkers, the average distance between the walkers is small. Thus, the average time of their contacts

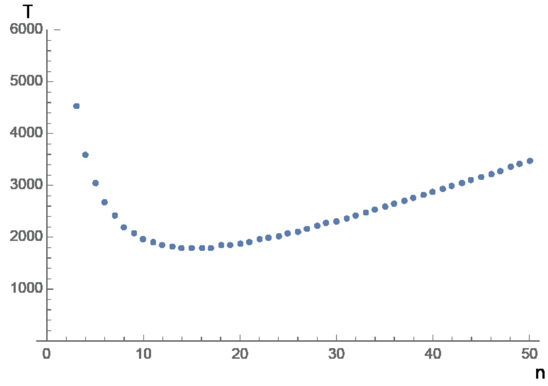

Fig. 5. Total relaying time T as a function of the number of walkers n when there is a delay of $d = 64$.

is short. If the delay time is longer than this average contact time, the total relaying time is prolonged. When the number of walkers is small, the delay time is more likely shorter than the average contact time. In this case, the total relaying time is not affected. The optimal number of walkers to have the shortest relaying time is in between.

We can roughly estimate this optimal number of walkers. The average contact time between the two walkers is roughly the square of the average distance N/n. If we set n^* as the optimal number of the walkers, the following relation is obtained.

$$d \approx (\frac{N}{n^*})^2$$

or

$$n^* \approx \frac{N}{\sqrt{d}}$$

In Figure 5, we see the minimum relaying time occurs at around $n \approx 15$. The estimation from the above equation is $n^* \approx 13$. Our preliminary simulation results indicate that this crude estimation is justified to a certain degree also with other parameter values.

There have been investigations of "Stochastic Resonance", which is a resonant-like behavior with an optimally tuned level of noise and oscillatory signals(2; 3; 5; 20). When we replace oscillatory signals by oscillation due to delay, we can still obtain a similar behavior. This is called "Delayed Stochastic Resonance", and has been studied both theoretically and experimentally(11; 14; 19). The DDR can be viewed as a collective version of Delayed Stochastic Resonance models.

4. CONCLUSION

We have presented two models which show interplays between stochasticity and delay. On Delayed Gambler's Ruin, exact analysis of the ruin probability is difficult when there is a difference between delays associated with gain and loss. We proposed an approximation scheme. The scheme essentially finds shifts in the initial assets to account for the effects of the delays and reduces the problem to a normal gambler's ruin with a shifted initial assets. Through computer simulations, we found that our approximation may account well for small delay differences, particularly for the case that the initial asset is closer to the mid-points between two boundaries. Further analysis is left for the future. Also, we may extend this model to mutually interacting multiple gamblers' ruin problems reflecting company bankruptcies in reality.

The Delayed Random Relays may find applications in transfer processes involving multiple entities. For example, the sun generates its thermal energy at its core, but it takes unusually long time for the energy to go through the radiative zone to reach the sun's surface. This time ranges are estimated from 170,000 to 50,000,000 years, but the detailed mechanism is still unknown. Though quite rough, we may be able to estimate the delay in micro-transfer of energy among the particles involved, if we can estimate the number of particles or the density of various parts of the sun. Another area of application is an epidemic spreading with latent periods. By observing spreading speed and number of infected in a large scale, one may infer the latent periods. Qualitatively similar behavior may arise in rumor spreadings or in passing down of folklore stories.

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