Stability conditions for time delay systems in terms of the Lyapunov matrix ⋆

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Abstract: An overview of stability conditions in terms of the Lyapunov matrix for time-delay systems is presented. The main results and proof strategies are outlined in the retarded type case. The state of the art and potential extensions to other classes of delay systems are discussed.

Keywords: Delay systems; Stability criterion; delay Lyapunov matrix.

1. INTRODUCTION

The main reason of the success of Lyapunov functions methods is probably that they allows the investigation of the stability of dynamical systems without knowing the solution of the equations governing the systems. This outstanding feature is evident in the case of linear delay free systems where the positivity of the so-called Lyapunov matrix is a stability criterion (i.e., necessary and sufficient condition).

More precisely, the proof that the linear system

$$\dot{x} = Ax$$

is stable if and only if there exist a function of the form $v(x) = x^T Px$ that satisfies $v(x) > 0$, $x \neq 0$, and $\frac{dv(x)}{dt} < 0$, $x \neq 0$, along the trajectories of the system established in the framework of the Lyapunov framework reduces, when the Lyapunov condition is satisfied, to solving for any positive definite matrix Q, the algebraic equation

$$A^T P + PA = -Q$$

for the unknown matrix P, and to verify its positivity.

The generalization to time-delay systems by (Krasovskii, 1956) has led to the proposal of Lyapunov-Krasovskii functionals leading to sufficient stability conditions in the form of linear matrix inequalities. These results range from the early proposals of Kolmanovskii and Myshkis (1999), Niculescu (2001), to refined conditions such as those of Fridman (2014), Seuret and Gouaisbaut (2015), among many others.

The converse results of the theory of Krasovskii, that guarantee the existence of the functional when the system is stable has been less popular, although the general form of the functional introduced by Repin (1965) and Datko (1972), has been a source of inspiration for the determination of sufficient stability conditions in Gu (2001), Peet and Bliman (2011).

In the past decades, following the early works of Infante and Castelan (1978), Huang (1989), Louisell (2001) in this direction, a comprehensive coverage of the case of linear delay systems of retarded, distributed and neutral type has been made by Kharitonov (2013). The core of the presented results include the functional, which is obtained from the Cauchy formula and expressed in terms of the delay Lyapunov matrix, the construction of the Lyapunov matrix via the three properties called symmetry, dynamic and algebraic, and the proof of existence of a quadratic lower bound whenever the system is stable.

These results are indeed a nontrivial extension of the linear delay free case. They were successfully applied to problems such as robust stability analysis (Kharitonov and Zhabko, 2003), determination of exponential estimates (Kharitonov and Hinrichsen, 2004), computation of the $\mathcal{H}_2$ norm (Jarlebring et al., 2011), solution of the suboptimal control problem (Santos et al., 2009) and design of predictor-based control schemes for state and input delay systems (Kharitonov, 2014). A notable fact is that these applications share the common assumption that the underlying system is exponentially stable. This raises naturally the following question: *is it possible to asses the stability of the system in this framework? Moreover, considering the analogy with the delay free case, does a stability criterion expressed in terms of the delay Lyapunov matrix exists*?

The purpose of this paper is to summarize the advances in the answer to this query. We introduce the basic definitions and stability theorems for the multiple delay case in Section 2. We outline the path to our stability conditions and prove the necessity in Section 3. We present an infinite and finite criterion in Section 4 and Section 5, respectively. Finally, we give a brief overview of current advances in Section 6 and outline possible extensions in Section 7.

Notation: The space of piecewise continuous and continuously differentiable vector functions on $[-H, 0]$ is represented by $PC\left([-H, 0], \mathbb{R}^n\right)$ and $C^{(1)}\left([-H, 0], \mathbb{R}^n\right)$, respectively. The vector Euclidean norm and the matrix induced norm are both denoted by $\|\cdot\|$. For functions $\varphi \in PC([-H, 0], \mathbb{R}^n)$, we use the uniform norm

$$\|\varphi\|_H = \sup_{\theta\in[-H,0]} \|\varphi(\theta)\|$$

⋆ Partially supported by Project Conacyt 180725.

and the seminorm

$$\|\varphi\|_{\mathcal{H}} = \sqrt{\|\varphi(0)\|^2 + \int_{-H}^{0} \|\varphi(\theta)\|^2 \, d\theta}.$$

The notation $A > 0$ ($A \geqslant 0$, $A \ngeqslant 0$) means that the symmetric matrix A is positive definite (positive semidefinite, not positive semidefinite). The square block matrix with i-th row and j-th column element A_{ij} is denoted $[A_{ij}]_{i,j=1}^{r}$. The minimum eigenvalue of a matrix A is represented by $\lambda_{\min}(A)$. The function that maps y to the least integer greater or equal to y is denoted by $\lceil y \rceil$.

2. PRELIMINARIES

2.1 Basic system definitions

Consider a linear system of the form

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t - h_j), \quad t \geqslant 0, \tag{1}$$

where $A_0, \ldots, A_m$ are constant real $n \times n$ matrices, and $0 = h_0 < h_1 < \ldots < h_m = H$ are the delays.

Without any loss of generality, the initial time instant can be set equal to zero. The initial functions φ are taken from the space of piecewise continuous functions $PC([-H, 0], \mathbb{R}^n)$. The restriction of the solution $x(t, \varphi)$ of system (1) on the interval $[t - H, t]$ is denoted by

$$x_t(\varphi) : \theta \to x(t + \theta, \varphi), \quad \theta \in [-H, 0].$$

Definition 1. System (1) is said to be *exponentially stable*, if there exist constants $\gamma \geqslant 1$ and $\sigma > 0$, such that

$$\|x(t, \varphi)\| \leqslant \gamma e^{-\sigma t} \|\varphi\|_{H}, \quad t \geqslant 0.$$

The matrix-function $K(t)$, satisfying the equation

$$\dot{K}(t) = \sum_{j=0}^{m} A_j K(t - h_j), \quad t \geqslant 0,$$

with the initial conditions

$$K(0) = I, \quad K(t) = 0, \ t < 0,$$

is called the *fundamental matrix* of system (1). It allows an expression of the solution on $[0, \infty)$ via the Cauchy formula:

$$x(t, \varphi) = K(t)\varphi(0) + \sum_{j=1}^{m} \int_{-h_j}^{0} K(t - \theta - h_j) A_j \varphi(\theta) d\theta. \tag{2}$$

2.2 Converse approach framework

We remind now the main definitions and results of the converse approach. According to Kharitonov and Zhabko (2003), for any positive definite matrix W, the functional satisfying

$$\frac{dv_0(x_t(\varphi))}{dt} = -x^T(t, \varphi) W x(t, \varphi)$$

along the solutions of system (1), has the form

$$\begin{aligned} v_0(\varphi) &= \varphi^T(0) U(0) \varphi(0) + 2\varphi^T(0) \cdot \\ &\cdot \sum_{j=1}^{m} \int_{-h_j}^{0} U^T(\theta + h_j) A_j \varphi(\theta) d\theta + \sum_{k=1}^{m} \int_{-h_k}^{0} \varphi^T(\theta_1) A_k^T \cdot \\ &\cdot \sum_{j=1}^{m} \int_{-h_j}^{0} U(\theta_1 + h_k - \theta_2 - h_j) A_j \varphi(\theta_2) \, d\theta_2 \, d\theta_1, \end{aligned} \tag{3}$$

where the matrix-valued continuous function

$$U(\tau) = \int_{0}^{\infty} K^T(t) W K(t + \tau) dt, \ \tau \in \mathbb{R}, \tag{4}$$

is the *the delay Lyapunov matrix.* This definition requires the exponential stability assumption of the system in order to ensure convergence of the integral. This is overcome in the next definition.

Definition 2. (Kharitonov (2013)). The delay Lyapunov matrix $U(\tau)$, $\tau \in \mathbb{R}$, of system (1), associated with a given symmetric matrix W, is a continuous function satisfying the following three properties:

$$U'(\tau) = \sum_{j=0}^{m} U(\tau - h_j) A_j, \quad \tau \geqslant 0, \tag{5}$$

$$U(\tau) = U^T(-\tau), \quad \tau \in \mathbb{R}, \tag{6}$$

$$\sum_{j=0}^{m} \left(U(-h_j) A_j + A_j^T U(h_j) \right) = -W. \tag{7}$$

Matrix $U(\tau)$ exists and is unique if the Lyapunov condition holds (i. e., system (1) has no eigenvalues that are symmetric with respect to zero). It can be constructed via the semianalytic procedure (Kharitonov, 2013) or some numerical methods (Jarlebring et al., 2011; Huesca et al., 2009; Kharitonov, 2013; Egorov and Kharitonov, 2016), using the so-called dynamic (5), symmetry (6) and algebraic (7) properties, which play the role of the Lyapunov equation in the delay free case.

We introduce now the quadratic functional

$$\begin{aligned} v_1(\varphi) =& v_0(\varphi) + \int_{-H}^{0} \varphi^T(\theta) W \varphi(\theta) \, d\theta \\ =& \int_{-H}^{\infty} x^T(t, \varphi) W x(t, \varphi) dt, \end{aligned} \tag{8}$$

whose derivative along the solutions of system (1) is

$$\frac{dv_1(x_t(\varphi))}{dt} = -x^T(t - H, \varphi) W x(t - H, \varphi).$$

We formulate the following stability theorem:

Theorem 3. If system (1) is exponentially stable, then there exists $\alpha_1 > 0$, such that

$$v_1(\varphi) \geqslant \alpha_1 \|\varphi\|_{\mathcal{H}}^2, \quad \varphi \in PC([-H, 0], \mathbb{R}^n). \tag{9}$$

From the previous theorem, one deduce the lower bound

$$v_1(\varphi) \geqslant \alpha_1^{\star} \|\varphi(0)\|^2, \quad \varphi \in PC\left([-H, 0], \mathbb{R}^n\right).$$

Here, $\alpha_1^{\star}$ is a real positive number that can be computed whether the system is stable or not. See, for instance, Theorem 1 in (Egorov, 2016).

3. NECESSARY STABILITY CONDITIONS

3.1 The path to the necessary stability conditions

The scalar one delay equations stability criteria in Mondié (2012) and Egorov and Mondié (2013) indicated that there was hope in finding necessary and sufficient stability conditions for broader classes of systems. However the employed proof technique consisting on the exhaustive verification in the space of parameters of the proposed conditions with those known from frequency domain technique were unfit for more complex systems.

It appeared in the early attempts reported in Mondié et al. (2011) that the substitution of simple choices of the initial function into the expression of the functional, combined with the quadratic lower bound, lead to a variety of necessary stability conditions. For example,

$$\hat{\varphi}(\theta) = \begin{cases} \mu, & \theta = 0, \\ 0, & \text{elsewhere}, \end{cases}$$

where μ is an arbitrary non-zero vector, is such that $v_1(\hat{\varphi}) = \mu^T U(0)\mu$. Then, the lower bound (9) and the symmetry property of $U(0)$ imply that a necessary stability condition of the system is $U(0) > 0$. A trial and error approach lead to discover in Mondié et al. (2012) that, in the multivariable one delay case, initial functions of the form $\mu e^{A_0\theta}$, where μ is an arbitrary non-zero vector and A_0 is the matrix parameter of the system, allowed to put into evidence the dependence on the delay Lyapunov matrix. A decisive step was the observation that in this case $e^{A_0\theta}$ was indeed the expression of the fundamental matrix $K(\theta)$, $\theta \in [0, h]$. As a result of this observation, the following initial function was introduced in Egorov and Mondié (2014) for addressing the multiple delay case:

$$\psi_r(\theta) = \sum_{i=1}^{r} K(\theta + \tau_i)\gamma_i, \quad \theta \in [-H, 0], \tag{10}$$

where $\tau_i \in [0, H]$ and $\gamma_i \in \mathbb{R}^n$. The key role that the function ψ_r plays is described in more detail next.

3.2 A direct approach

We present a strategy that we call "direct approach" that uncovers the form of the necessary stability conditions in terms of the delay Lyapunov matrix, and that can be easily extended to other classes of systems. However, this section relies on the stability assumption of the system and cannot be used in order to prove the stability criterion.

The following result stems from the Cauchy formula for the fundamental matrix, as every column of $K(t + \tau)$ is a solution of (1) for any $\tau \geqslant 0$.

Lemma 4. For $t \geqslant 0$ and $\tau \geqslant 0$, the fundamental matrix satisfies

$$\begin{aligned} &K(t + \tau_i) = \\ &= K(t)K(\tau_i) + \sum_{j=1}^{m} \int_{-h_j}^{0} K(t - \theta - h_j)A_j K(\theta + \tau_i)d\theta. \end{aligned}$$

Consider (10) as initial function. From equation (2),

$$\begin{aligned} x(t, \psi_r) = \sum_{i=1}^{r} \Bigg(&K(t)K(\tau_i) \\ &+ \sum_{j=1}^{m} \int_{-h_j}^{0} K(t - \theta - h_j)A_j K(\theta + \tau_i)d\theta \Bigg) \gamma_i. \end{aligned}$$

In view of Lemma 4, this is

$$x(t, \psi_r) = \sum_{i=1}^{r} K(t + \tau_i)\gamma_i, \; t \geqslant -H. \tag{11}$$

The *assumption of the exponential stability* of system (1) and the substitution of (11) into (8) yield

$$\begin{aligned} v_1(\psi_r) &= \int_{-H}^{\infty} x^T(t, \psi_r) W x(t, \psi_r) dt \\ &= \int_{-H}^{\infty} \left(\sum_{i=1}^{r} \gamma_i^T K^T(t + \tau_i) \right) W \left(\sum_{i=1}^{r} K(t + \tau_i)\gamma_i \right) dt \\ &= \sum_{j=1}^{r} \sum_{i=1}^{r} \gamma_i^T \int_{-H}^{\infty} K^T(t + \tau_i) W K(t + \tau_j) dt \gamma_j, \end{aligned}$$

and the change of variable $t + \tau_i = s$ gives

$$\begin{aligned} v_1(\psi_r) &= \sum_{j=1}^{r} \sum_{i=1}^{r} \gamma_i^T \left(\int_{\tau_i - H}^{\infty} K^T(s) W K(s + \tau_j - \tau_i) ds \right) \gamma_j \\ &= \sum_{j=1}^{r} \sum_{i=1}^{r} \gamma_i^T \left(\int_{0}^{\infty} K^T(s) W K(s + \tau_j - \tau_i) ds \right) \gamma_j. \end{aligned}$$

By using equality (4), we get

$$v_1(\psi_r) = \sum_{j=1}^{r} \sum_{i=1}^{r} \gamma_i U(\tau_j - \tau_i)\gamma_j = \gamma^T \left[U(\tau_j - \tau_i) \right]_{i,j=1}^{r} \gamma,$$

where $\gamma = \begin{pmatrix} \gamma_1^T & \dots & \gamma_r^T \end{pmatrix}^T$. This fact and Theorem 3 suggest that

$$\left[U(\tau_j - \tau_i) \right]_{i,j=1}^{r} \geqslant 0,$$

hence we can discern that the necessary stability condition of Theorem 3 can be written in terms of the Lyapunov matrix, as in the case of delay free systems.

3.3 Necessary stability conditions via new properties of the Lyapunov matrix

When seeking necessary and sufficient stability conditions, the stability assumption is clearly ruled out. Instead of using the integral form of functional (8), as in the previous section, we must use expression (3) given in terms of the delay Lyapunov matrix, which is valid whether the system is stable or not. The substitution of the initial function ψ_r given by (10), results in products of the fundamental and the delay Lyapunov matrices. This interplay is described by a number of properties based on Definition 2 in Egorov and Mondié (2014). They are recalled next.

The first property, called the generalized algebraic property, is determined for $\tau \geqslant 0$ by

$$\sum_{j=0}^{m} \left(U(\tau - h_j)A_j + A_j^T U(\tau + h_j) \right) = -W K(\tau).$$

One can see that it reduces to the algebraic property (7) when $\tau = 0$. The next one is given for $\tau_1 \geqslant 0$, $\tau_2 \in \mathbb{R}$ by

$$\begin{aligned} U(\tau_1 + \tau_2) = &\; U(\tau_2)K(\tau_1) \\ &+ \sum_{j=1}^{m} \int_{-h_j}^{0} U(\tau_2 - \theta - h_j)A_j K(\tau_1 + \theta)\, d\theta \\ &+ \int_{-\tau_1}^{0} K^T(\tau_2 + \theta) W K(\tau_1 + \theta)\, d\theta. \end{aligned}$$

A key element of our approach is the bilinear functional

$$z(\varphi, \psi) = \frac{v_1(\varphi + \psi) - v_1(\varphi - \psi)}{4} =$$

$$= \varphi^T(0)U(0)\psi(0)$$
$$+ \varphi^T(0)\sum_{j=1}^{m}\int_{-h_j}^{0} U^T(\theta + h_j)A_j\psi(\theta)d\theta$$
$$+ \sum_{j=1}^{m}\int_{-h_j}^{0} \varphi^T(\theta)A_j^T U(\theta + h_j)\, d\theta\psi(0) + \sum_{k=1}^{m}\int_{-h_k}^{0} \varphi^T(\theta_1)\, \cdot$$
$$\cdot\ A_k^T \sum_{j=1}^{m}\int_{-h_j}^{0} U(\theta_1 + h_k - \theta_2 - h_j)A_j\psi(\theta_2)d\theta_2 d\theta_1$$
$$+ \int_{-H}^{0} \varphi^T(\theta)W\psi(\theta)\, d\theta.$$

When introducing in this functional the initial functions

$$\varphi(\theta) = K(\tau_1 + \theta)\mu,\ \ \psi(\theta) = K(\tau_2 + \theta)\eta,\ \theta \in [-H, 0],$$

with τ_1, $\tau_2 \in [0, H]$ and μ, η arbitrary non-zero vectors, the instrumental properties introduced above allow the following striking reduction: For any τ_1, $\tau_2 \in [0, H]$,

$$z\left(K(\tau_1 + \theta)\mu, K(\tau_2 + \theta)\eta\right) = \mu^T U(-\tau_1 + \tau_2)\eta.$$

The previous equality enables us to prove the necessary conditions for the exponential stability of system (1).

Theorem 5. If system (1) is exponentially stable, then

$$\widehat{\mathcal{K}}_r(\tau_1, \ldots, \tau_r) := \left[U(-\tau_i + \tau_j)\right]_{i,j=1}^{r} > 0, \qquad (12)$$

where $\tau_k \in [0, H]$, $k = \overline{1, r}$, and $\tau_i \neq \tau_j$, if $i \neq j$.

Remark 6. Theorem 5 provides a family of necessary stability conditions, whose complexity increases with the parameter r in (12). It is worth mentioning that the case $r = 1$ reduces to the simplest condition $U(0) > 0$, which is necessary and sufficient for the exponential stability of the delay free system, and that for $r = 2$ the stability criterion for the single delay scalar equation is recovered.

4. INFINITE STABILITY CRITERION

The main result of this section, a stability criterion for systems with pointwise and distributed delays, is proved in full detail in Egorov (2014) and Egorov et al. (2017).

We recall first the following instability result, based on the one proven in Medvedeva and Zhabko (2015).

Theorem 7. If system (1) is unstable and satisfies the Lyapunov condition, then for every $\alpha_1 > 0$ there exists a function $\widehat{\varphi} \in C^{(1)}([-H, 0], \mathbf{R}^n)$ such that $v_1(\widehat{\varphi}) \leqslant -\alpha_1$.

A key result in the proof is that it is possible to approximate any continuously differentiable function by a function of the form (10) with equidistant points τ_i.

Lemma 8. For any $\varphi \in C^{(1)}([-H, 0], \mathbb{R}^n)$ and any $\varepsilon > 0$, there exists a function ψ_r of the form (10) with

$$\tau_i = (i-1)\delta_r,\ \ i = \overline{1, r},\ \ \delta_r = \frac{1}{r-1}H, \qquad (13)$$

such that $\|\varphi - \psi_r\|_{\mathcal{H}} < \varepsilon$.

Then, by introducing the matrix

$$\mathcal{K}_r = \widehat{\mathcal{K}}_r\left(0, \frac{1}{r-1}H, \ldots, \frac{r-2}{r-1}H, H\right),$$

and using Theorem 7 and Lemma 8, it is shown that the necessary stability condition of Theorem 5 is also sufficient for large enough r.

Theorem 9. System (1) is exponentially stable if and only if the Lyapunov condition holds and for every natural number r

$$\mathcal{K}_r > 0. \qquad (14)$$

Moreover, if the Lyapunov condition holds and system (1) is unstable, then there exists a natural number r such that

$$\mathcal{K}_r \not> 0.$$

In order to use Theorem 9 for determining the stability of the system, one requires an *infinite* number of mathematical operations, as condition (14) demands to be tested for every natural r. Although the theorem states that for every unstable system there exists a number r such that condition (14) does not hold, it does not provide any estimate of such number.

5. FINITE STABILITY CRITERION

In this section, we present the *finite* stability criterion introduced in Egorov (2016) for systems of the form (1). The following auxiliary results are instrumental in the proof.

Lemma 10. There exists a number $\alpha_2 > 0$ such that

$$|z(\varphi, \psi)| \leqslant \alpha_2 \|\varphi\|_{_H} \|\psi\|_{_H},$$
$$|v_1(\varphi)| \leqslant \alpha_2 \|\varphi\|_{_H}^2,$$

for any $\varphi, \psi \in PC\left([-H, 0], \mathbb{R}^n\right)$.

Consider now the compact set

$$\mathcal{S} = \{\varphi \in C^{(1)}\left([-H, 0], \mathbb{R}^n\right) : \|\varphi\|_{_H} = \|\varphi(0)\| = 1, \|\varphi'\| \leqslant M\},$$

where $M = \sum_{i=0}^{m} \|A_i\|$. The following theorem relates the function v_1 with functions from the set $\mathcal{S}$:

Theorem 11. System (1) is exponentially stable if and only if the Lyapunov condition holds and there exists a positive number α_1 such that for any $\varphi \in \mathcal{S}$, $v_1(\varphi) \geqslant \alpha_1$.

For a fixed $\varphi \in \mathcal{S}$, we construct the function ψ_r in (10) as:

(1) Set τ_1, …, τ_r, as in (13).
(2) Choose the vectors γ_i, $i = \overline{1, r}$, such that
$$\varphi(-\tau_i) = \psi_r(-\tau_i),\ i = \overline{1, r}.$$

The next lemma shows that every function from the set $\mathcal{S}$ can be approximated by a function ψ_r constructed by the previous steps. Unlike in Lemma 8, here, an approximation error bound is provided.

Lemma 12. For every $\varphi \in \mathcal{S}$,

$$\|\varphi - \psi_r\|_{_H} \leqslant \frac{(M+L)e^{LH}}{1/\delta_r + L},$$

where L is such that $\|K'(t)\| \leqslant L$, $t \in (0, H)$.

The finite stability criterion is established next with the help of Theorem 11 and Lemma 12.

Theorem 13. System (1) is exponentially stable if and only if the Lyapunov condition holds and

$$\mathcal{K}_r - \alpha_1 \mathcal{A}_r > 0,$$

where

$$\mathcal{A}_r = \left[K^T(\tau_i)K(\tau_j)\right]_{i,j=1}^{r},$$
$$r = 1 + \left\lceil He^{LH}(M+L)\left(\alpha + \sqrt{\alpha(\alpha+1)}\right) - HL\right\rceil,$$

and $\alpha = \alpha_2/\alpha_1$ and $\alpha_1 \in (0, \alpha_1^\star)$.

In Theorem 13 the stability criterion now also depends on the fundamental matrix. While it is true that the number r can be very large, a fact of theoretical significance is that one can determine the stability of the system with a finite number of mathematical operations. It is worth mentioning that in illustrative examples in Cuvas et al. (2017), the stability region is achieved with small r.

6. OVERVIEW OF THE STATE OF THE ART

We have been able to use the above approach to present necessary and in some cases necessary and sufficient stability conditions for a number of classes of systems with delays. We outline next the progress made up to now as well as ongoing research.

6.1 Systems with pointwise and distributed delays

The above results are extended with no difficulties in Cuvas and Mondié (2016) and Egorov et al. (2017) to the case of systems with distributed delays of the form

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t - h_j) + \int_{-H}^{0} G(\theta) x(t + \theta) d\theta,$$

where $G \in PC\left([-H, 0], \mathbb{R}^{n \times n}\right)$.

6.2 Systems of neutral type

We also have obtained in (Gomez et al., 2017b) necessary stability conditions of the same kind for systems of neutral type equations with multiple commensurate delays:

$$\frac{d}{dt}\left(\sum_{i=0}^{m} D_i x(t - ih)\right) = \sum_{i=0}^{m} A_i x(t - ih),$$

where $D_0 = I$, D_i, A_i, $i = \overline{0, m}$, belong to $\mathbb{R}^{n \times n}$, and $h > 0$ is the basic delay. They generalize those first obtained for the single delay case (Gomez et al. (2017a)). The neutral case is, as usual, more tricky, and special care has to be paid to the jump discontinuities of the system. Some advances on the finite stability criterion for systems with one delay are presented in Gomez et al. (2018) for the case in which the matrix D satisfies $\|D\| < 1$.

6.3 Linear periodic systems with delays

In Gomez et al. (2016), inspired by the developments of converse results presented in Letyagina and Zhabko (2009), we study systems described by

$$\dot{x}(t) = \sum_{j=0}^{m} A_j(t) x(t - h_j), \tag{15}$$

where $A_j(t)$, $j = \overline{0, m}$, are matrices of continuous coefficients with period T, i.e., $A_j(t) = A_j(t + T)$, with range in $\mathbb{R}^{n \times n}$, and $0 = h_0 < h_1 < \ldots < h_m = H$ are constant delays. The stability conditions are similar to those obtained in the time-invariant case.

Theorem 14. Let system (15) be exponentially stable, then, the following condition holds for any positive integer r, $\tau_i \in [0, H]$, $i = \overline{1, r}$, and $t_0 \in [0, T)$:

$$\widehat{\mathcal{K}}_r(t_0, \tau_1, \ldots, \tau_r) = \left[U(t_0 - \tau_i, t_0 - \tau_j)\right]_{i,j=1}^{r} \geqslant 0.$$

In this case, the bi-dimensional delay Lyapunov matrix is the solution of a partial differential system with boundary conditions, which makes its computation a challenging issue that is under investigation.

6.4 Integral and continous time difference equations

In our recent work, we have applied the so-called direct approach to non-differential linear systems, namely, difference equations of the form

$$x(t) = \sum_{j=1}^{m} A_j x(t - h_j),$$

where $x(t) \in \mathbb{R}^n$, A_1, $\ldots$, A_m are constant real $n \times n$ matrices, and $0 = h_0 < h_1 \cdots < h_m = H$ are the delays, and also to integral equations of the form

$$x(t) = \int_{-H}^{0} G(\theta) x(t + \theta) d\theta,$$

with kernel $G \in PC\left([-H, 0], \mathbb{R}^{n \times n}\right)$, and delay H. We have used the so-called direct approach to determine successfully the form of the necessary stability conditions in Rocha et al. (2016) and del Valle et al. (2018), namely, for any $\tau_i \in [0, H]$, $i = \overline{1, r}$, $\tau_i \neq \tau_j$, if $i \neq j$,

$$\widehat{\mathcal{K}}_r(\tau_1, \ldots, \tau_r) = \left[F(\tau_i, \tau_j)\right]_{i,j=1}^{r} > 0,$$

where $F(\tau_i, \tau_j) = U(0) - U(-\tau_i) - U(\tau_j) + U(\tau_j - \tau_i)$. In the case of the difference equation in continuous time, we have introduced the Dini upper right-hand derivative of a functional to address the jump discontinuities inherent to this class of systems. The obtained conditions are different from those for differential systems. It may be possible to obtain conditions in the simpler form (12).

7. FUTURE DIRECTIONS OF RESEARCH

As a conclusion, we would like to outline future directions of research in the area. First, the remaining gaps in the sufficiency should be filled by addressing the particular issues raised by each case. Second, the complexity in the numerical construction of the delay Lyapunov matrix deserves attention, as the NP-hard nature of delay systems resurfaces in this task. Indeed, the applicability of the conditions highly depends on this construction. Finally, we believe that it is possible to use our approach to find necessary/necessary and sufficient stability conditions for other classes of linear systems, such as those of fractional order, stochastic systems, or some classes of partial differential equations, by developing first the Cauchy formula, the fundamental and Lyapunov matrix definition, and the corresponding stability theorems of the converse approach.

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