On Hamilton-Jacobi equations for neutral-type differential games ⋆

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Abstract: We consider a two-person zero-sum differential game in which a motion of the dynamical system is described by neutral-type functional-differential equations in Hale's form and the quality index estimates a motion history realized up to the terminal instant of time and includes integral estimations of control realizations of the players. The formalization of the game in the class of pure positional strategies is given, the corresponding notions of the value functional and optimal control strategies of the players are defined. For the value functional, we derive a Hamilton-Jacobi type equation with coinvariant derivatives. It is proved that, if a solution of this equation satisfies certain smoothness conditions, then it coincides with the value functional. On the other hand, it is proved that, at the points of coinvariant differentiability, the value functional satisfies the derived Hamilton-Jacobi equation. Therefore, this equation can be called the Hamilton-Jacobi-Bellman-Isaacs equation for neutral-type systems.

Keywords: neutral-type system, differential game, Hamilton-Jacobi equation, coinvariant derivatives, value functional, optimal strategies.

1. INTRODUCTION

The paper is devoted to the development of differential games theory (see, e.g., Isaacs (1965); Krasovskii and Subbotin (1988); Osipov (1971)) and the corresponding Hamilton-Jacobi (HJ) equations (see, e.g., Subbotin (1995); Crandall and Lions (1983); Clarke, Ledyaev, Stern and Wolenski (1998); Lukoyanov (2000, 2003)) for functional-differential neutral-type systems.

For a dynamical system described by neutral-type functional-differential equations in Hale's form (see Hale and Cruz (1970)), a two-person zero-sum differential game is considered. The optimized quality index of the control process consists of two terms. The first one estimates the motion history realized up to the terminal instant of time. The second one contains an integral estimation of control realizations of players. Within the game-theoretical approach of Krasovskii and Subbotin (1988); Krasovskii and Krasovskii (1995), the differential game is formalized in the classes of pure positional strategies (see also Gomoyunov, Lukoyanov and Plaksin (2017)). Basing on the notion of coinvariant derivatives (see Kim (1999)), for the value functional of this differential game, a HJ type equation is derived. The main difference between this equation and the HJ equation for retarded functional-differential systems (see Lukoyanov (2000, 2003)) is the presence of a new term. It leads to difficulties in the analysis of the derived HJ equation and motivates its study. It is proved that a solution of this equation, satisfying certain smoothness conditions, is the value functional of the initial differential game. Moreover, it is shown that the players' strategies constructed by the extremal shift method in the direction of the coinvariant gradient of this functional are optimal. Using an appropriate notion of characteristic complexes (see, e.g., Subbotin (1995); Lukoyanov (2000)), it is proved that, at the points of coinvariant differentiability, the value functional satisfies the derived HJ equation. Thus, by analogy with Krasovskii and Subbotin (1988); Subbotin (1995); Lukoyanov (2003), this equation can be considered as the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation for neutral-type systems.

2. DIFFERENTIAL GAME

We consider a two-person zero-sum differential game for the dynamical system described by the neutral-type functional-differential equation in Hale's form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(x[t] - g\big(t, x_t[\cdot]\big)\Big) = f\big(t, x_t[\cdot], u[t], v[t]\big), \qquad t \in [t_0, \vartheta], \quad x[t] \in \mathbb{R}^n, \quad u[t] \in \mathbb{U}, \quad v[t] \in \mathbb{V}, \tag{1}$$

and the quality index

$$\gamma = \sigma\big(x_\vartheta[\cdot]\big) + \int_{t_*}^{\vartheta} \chi\big(\xi, x_\xi[\cdot], u[\xi], v[\xi]\big)\,\mathrm{d}\xi. \tag{2}$$

Here t is the time variable; $x[t]$ is the value of the state vector at the time t; t_0 and ϑ are fixed instants of time; $t_* \in [t_0, \vartheta]$ is the instant of the control process beginning; $h > 0$ is the delay constant; $x_t[\cdot]$ is the motion history on the interval $[t-h, t]$ defined by $x_t[\xi] = x[t+\xi]$, $\xi \in [-h, 0]$; $u[t]$ and $v[t]$ are control actions of the first

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and the second players, respectively; $\mathbb{U}$ and $\mathbb{V}$ are known compact subsets of finite-dimensional spaces. The first player aims to minimize the value γ of the quality index, while the second player aims to maximize it.

Below, we denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the Euclidean norm and the scalar product of vectors, respectively. Also, we denote by $\mathrm{Lip} = \mathrm{Lip}([-h,0],\mathbb{R}^n)$ the space of all Lipschitz continuous functions from $[-h,0]$ to $\mathbb{R}^n$ endowed with the supremum norm $\|\cdot\|_\infty$. For a number $\nu > 0$, we define

$$D_\nu := \big\{w[\cdot] \in \mathrm{Lip}\colon \big\|w[\cdot]\big\|_\infty \le \nu, \\ \big\|w[\xi'] - w[\xi'']\big\| \le \nu\big|\xi' - \xi''\big|, \xi', \xi'' \in [-h,0]\big\}. \quad (3)$$

Note that D_ν is a compact subset of Lip.

A pair $(t, w[\cdot]) \in [t_0,\vartheta] \times \mathrm{Lip}$ is called a position of system (1). The set of all positions is denoted by $\mathbb{G} = [t_0,\vartheta] \times \mathrm{Lip}$.

It is assumed that the mappings $g\colon \mathbb{G} \mapsto \mathbb{R}^n$, $f\colon \mathbb{G} \times \mathbb{U} \times \mathbb{V} \mapsto \mathbb{R}^n$, $\sigma\colon \mathrm{Lip} \mapsto \mathbb{R}$ and $\chi\colon \mathbb{G} \times \mathbb{U} \times \mathbb{V} \mapsto \mathbb{R}$ from (1), (2) are continuous and satisfy the following conditions:

(g) There exists $h_0 \in (0,h)$ such that, for any $\nu > 0$, there exists $\lambda_g > 0$ such that

$$\big\|g\big(t,w[\cdot]\big) - g\big(\tau,r[\cdot]\big)\big\| \\ \le \lambda_g\Big(\big|t-\tau\big| + \max_{\xi\in[-h,-h_0]}\big\|w[\xi] - r[\xi]\big\|\Big)$$

for all $(t,w[\cdot]), (\tau,r[\cdot]) \in [t_0,\vartheta] \times D_\nu$.

(f) There exists $\alpha_f > 0$ such that

$$\big\|f\big(t,w[\cdot],u,v\big)\big\| \le \alpha_f\big(1 + \big\|w[\cdot]\big\|_\infty\big)$$

for any $(t,w[\cdot]) \in \mathbb{G}$, $u \in \mathbb{U}$ and $v \in \mathbb{V}$.

$(f,\chi.1)$ For any $\nu > 0$, there exist $\lambda_f, \lambda_\chi > 0$ such that

$$\big\|f\big(t,w[\cdot],u,v\big) - f\big(t,r[\cdot],u,v\big)\big\| \le \lambda_f\big\|w[\cdot] - r[\cdot]\big\|_\infty, \\ \big|\chi\big(t,w[\cdot],u,v\big) - \chi\big(t,r[\cdot],u,v\big)\big| \le \lambda_\chi\big\|w[\cdot] - r[\cdot]\big\|_\infty$$

for any $(t,w[\cdot]), (t,r[\cdot]) \in [t_0,\vartheta] \times D_\nu$, $u \in \mathbb{U}$ and $v \in \mathbb{V}$.

$(f,\chi.2)$ For any $(t,w[\cdot]) \in \mathbb{G}$ and $s \in \mathbb{R}^n$, we have

$$\min_{u\in\mathbb{U}}\max_{v\in\mathbb{V}}\Big(\big\langle f\big(t,w[\cdot],u,v\big), s\big\rangle + \chi\big(t,w[\cdot],u,v\big)\Big) \\ = \max_{v\in\mathbb{V}}\min_{u\in\mathbb{U}}\Big(\big\langle f\big(t,w[\cdot],u,v\big), s\big\rangle + \chi\big(t,w[\cdot],u,v\big)\Big).$$

Let an initial position $(t_*, x_*[\cdot]) \in \mathbb{G}$, $t_* < \vartheta$ be chosen. By admissible control realizations of the first and the second players, we mean measurable functions $u[\cdot]\colon [t_*,\vartheta) \mapsto \mathbb{U}$ and $v[\cdot]\colon [t_*,\vartheta) \mapsto \mathbb{V}$, respectively. Under the conditions above, following, for example, the scheme from Filippov (1988) (see also Hale and Cruz (1970)), one can show that such realizations uniquely generate from the position $(t_*, x_*[\cdot])$ the motion $x[\cdot]\colon [t_* - h,\vartheta] \mapsto \mathbb{R}^n$ of system (1) that is an absolutely continuous function which satisfies $x_{t_*}[\cdot] = x_*[\cdot]$ and, together with $u[\cdot]$ and $v[\cdot]$, satisfies equation (1) almost everywhere on $[t_*,\vartheta]$. The triple $\{x[\cdot], u[\cdot], v[\cdot]\}$ is called a control process realization. Note that this control process realization uniquely defines the value of quality index (2).

According to Krasovskii and Subbotin (1988); Krasovskii and Krasovskii (1995) (see also Gomoyunov, Lukoyanov and Plaksin (2017) for neutral-type systems), differential game (1), (2) is posed as follows.

By a control strategy of the first player, we mean an arbitrary function $U\colon \mathbb{G} \mapsto \mathbb{U}$. Let us fix an initial position $(t_*, x_*[\cdot]) \in \mathbb{G}$ and a partition of the interval $[t_*,\vartheta]$:

$$\Delta_\delta = \big\{\tau_j\colon \tau_0 = t_*, 0 < \tau_j - \tau_{j-1} \le \delta, j = \overline{1,l}, \tau_l = \vartheta\big\}. \quad (4)$$

The pair $\{U, \Delta_\delta\}$ defines a control law that forms a piecewise constant (and therefore, admissible) realization $u[\cdot]$ according to the following step-by-step rule:

$$u[t] = U(\tau_j, x_{\tau_j}[\cdot]), \quad t \in [\tau_j, \tau_{j+1}), \quad j = \overline{0, l-1}. \quad (5)$$

This control law together with an admissible control realization $v[\cdot]$ of the second player uniquely generate the control process realization $\{x[\cdot], u[\cdot], v[\cdot]\}$ and define the value $\gamma = \gamma(t_*, x_*[\cdot]; U, \Delta_\delta; v[\cdot])$ of quality index (2).

The guaranteed result of the strategy U is defined by

$$\rho_u\big(t_*, x_*[\cdot]; U\big) = \lim_{\delta\downarrow 0}\sup_{\Delta_\delta}\sup_{v[\cdot]}\gamma\big(t_*, x_*[\cdot]; U, \Delta_\delta; v[\cdot]\big). \quad (6)$$

The optimal guaranteed result of the first player is the following value:

$$\rho_u^\circ\big(t_*, x_*[\cdot]\big) = \inf_U \rho_u\big(t_*, x_*[\cdot]; U\big). \quad (7)$$

A strategy U° of the first player is called optimal if

$$\rho_u\big(t_*, x_*[\cdot]; U^\circ\big) = \rho_u^\circ\big(t_*, x_*[\cdot]\big).$$

Similarly, with the corresponding changes, for the second player, we define a control strategy $V\colon \mathbb{G} \mapsto \mathbb{V}$, control law $\{V, \Delta_\delta\}$ that forms a realization $v[\cdot]$ by

$$v[t] = V(\tau_j, x_{\tau_j}[\cdot]), \quad t \in [\tau_j, \tau_{j+1}), \quad j = \overline{0, l-1},$$

the guaranteed result of the strategy V

$$\rho_v\big(t_*, x_*[\cdot]; V\big) = \lim_{\delta\downarrow 0}\inf_{\Delta_\delta}\inf_{u[\cdot]}\gamma\big(t_*, x_*[\cdot]; u[\cdot]; V, \Delta_\delta\big), \quad (8)$$

and the optimal guaranteed result

$$\rho_v^\circ\big(t_*, x_*[\cdot]\big) = \sup_V \rho_v\big(t_*, x_*[\cdot]; V\big). \quad (9)$$

A strategy V° of the second player is called optimal if

$$\rho_v\big(t_*, x_*[\cdot]; V^\circ\big) = \rho_v^\circ\big(t_*, x_*[\cdot]\big).$$

Due to definitions (6)–(9), we have

$$\rho_v^\circ\big(t_*, x_*[\cdot]\big) \le \rho_u^\circ\big(t_*, x_*[\cdot]\big), \quad \big(t_*, x_*[\cdot]\big) \in \mathbb{G}. \quad (10)$$

If the equality $\rho^\circ\big(t_*, x_*[\cdot]\big) := \rho_u^\circ\big(t_*, x_*[\cdot]\big) = \rho_v^\circ\big(t_*, x_*[\cdot]\big)$ holds for any $(t_*, x_*[\cdot]) \in \mathbb{G}$, then $\rho^\circ\colon \mathbb{G} \mapsto \mathbb{R}$ is called the value functional of differential game (1), (2).

One can show (see, e.g., Krasovskii and Subbotin (1988); Krasovskii and Krasovskii (1995) and also Gomoyunov, Lukoyanov and Plaksin (2017)) that the value functional ρ° is continuous and has the following properties:

$(\rho.1)$ The following equality is valid:

$$\rho^\circ\big(\vartheta, x_*[\cdot]\big) = \sigma\big(x_*[\cdot]\big), \quad x_*[\cdot] \in \mathrm{Lip}.$$

$(\rho.2)$ For any $(t_*, x_*[\cdot]) \in \mathbb{G}$, $t^* \in [t_*,\vartheta]$, $\varepsilon > 0$ and any admissible realization $v[\cdot]$, there exists an admissible realization $u[\cdot]$ such that, for the corresponding control process realization $\{x[\cdot], u[\cdot], v[\cdot]\}$, we have

$$\rho^\circ\big(t^*, x_{t^*}[\cdot]\big) + \int_{t_*}^{t^*}\chi\big(\xi, x_\xi[\cdot], u[\xi], v[\xi]\big)\,\mathrm{d}\xi \le \rho^\circ\big(t_*, x_*[\cdot]\big) + \varepsilon.$$

$(\rho.3)$ For any $(t_*, x_*[\cdot]) \in \mathbb{G}$, $t^* \in [t_*,\vartheta]$, $\varepsilon > 0$ and any admissible realization $u[\cdot]$, there exists an admissible realization $v[\cdot]$ such that, for the corresponding control process realization $\{x[\cdot], u[\cdot], v[\cdot]\}$, we have

$$\rho^\circ\big(t^*, x_{t^*}[\cdot]\big) + \int_{t_*}^{t^*}\chi\big(\xi, x_\xi[\cdot], u[\xi], v[\xi]\big)\,\mathrm{d}\xi \ge \rho^\circ\big(t_*, x_*[\cdot]\big) - \varepsilon.$$

Note that, for similar differential games, the questions of existence of the value functional and players' optimal strategies are considered in Gomoyunov, Lukoyanov and Plaksin (2017); Gomoyunov and Lukoyanov (2018).

3. HJBI EQUATION

Let $(t, w[\cdot]) \in \mathbb{G}$. Denote

$$X\big(t, w[\cdot]\big) := \big\{x[\cdot] \in \mathrm{Lip}\big([t-h, \vartheta], \mathbb{R}^n\big) \colon x_t[\cdot] = w[\cdot]\big\}.$$

Following Kim (1999); Lukoyanov (2000), a functional $\varphi \colon \mathbb{G} \mapsto \mathbb{R}$ is called coinvariant differentiable (ci-differentiable) at a point $(t, w[\cdot])$ if there exist $\partial_t \varphi(t, w[\cdot]) \in \mathbb{R}$ and $\nabla\varphi(t, w[\cdot]) \in \mathbb{R}^n$ such that, for any $x[\cdot] \in X(t, w[\cdot])$, the following relation holds:

$$\varphi\big(\tau, x_\tau[\cdot]\big) - \varphi\big(t, w[\cdot]\big) = \partial_t \varphi\big(t, w[\cdot]\big)(\tau - t) + \langle x_\tau[0] - w[0], \nabla\varphi\big(t, w[\cdot]\big)\rangle + o(\tau - t), \quad \tau \in [t, \vartheta],$$

where $o(\tau - t)$ depends on the choice of the instant t and the function $x[\cdot]$, and $o(\tau-t)/(\tau-t) \to 0$ as $\tau \to t+0$. The values $\partial_t\varphi(t, w[\cdot])$ and $\nabla\varphi(t, w[\cdot])$ are called ci-derivatives of φ at the point $(t, w[\cdot])$ (also, $\nabla\varphi(t, w[\cdot])$ is called the ci-gradient of φ). One can show uniqueness of ci-derivatives.

Similarly, a mapping $\mathbb{G} \ni (t, w[\cdot]) \mapsto \psi = (\psi_1, ..., \psi_n) \in \mathbb{R}^n$ is called ci-differentiable at a point $(t, w[\cdot])$ if the functionals $\psi_i \colon \mathbb{G} \mapsto \mathbb{R}$, $i = \overline{1, n}$, are ci-differentiable at this point. In this case, we denote

$$\partial_t \psi\big(t, w[\cdot]\big) = \big(\partial_t \psi_1\big(t, w[\cdot]\big), \ldots, \partial_t \psi_n\big(t, w[\cdot]\big)\big),$$
$$\nabla\psi\big(t, w[\cdot]\big) = \big(\nabla\psi_1\big(t, w[\cdot]\big), \ldots, \nabla\psi_n\big(t, w[\cdot]\big)\big).$$

By the mapping g from (1), we introduce the set

$$\mathbb{G}_* = \big\{\big(t, w[\cdot]\big) \in \mathbb{G} \colon g \text{ is ci-differentiable at } \big(t, w[\cdot]\big) \text{ and } \nabla g\big(t, w[\cdot]\big) = 0\big\}. \quad (11)$$

The lemma below establishes some ci-differentiability properties of g.

Lemma 1. Let $(t_*, x_*[\cdot]) \in \mathbb{G}$ and $x[\cdot] \in X(t_*, x_*[\cdot])$. Then, for almost all $t \in [t_*, \vartheta]$, we have $(t, x_t[\cdot]) \in \mathbb{G}_*$ and

$$\partial_t g(t, x_t[\cdot]) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(g\big(t, x_t[\cdot]\big)\Big). \quad (12)$$

Proof. By (g), the function $a[t] := g(t, x_t[\cdot])$, $t \in [t_*, \vartheta]$, is Lipshitz continuous. Therefore, it is differentiable for almost all $t \in (t_*, \vartheta)$. Let $t \in (t_*, \vartheta)$ be such that $\mathrm{d}a[t]/\,\mathrm{d}t$ exists and $y[\cdot] \in X(t, x_t[\cdot])$. Define

$$\theta(\tau - t) := g\big(\tau, y_\tau[\cdot]\big) - g\big(t, x_t[\cdot]\big) - (\tau - t)\,\mathrm{d}a[t]/\,\mathrm{d}t, \ \tau \in [t, \vartheta].$$

From (g) it follows that

$$g\big(\tau, y_\tau[\cdot]\big) = g\big(\tau, x_\tau[\cdot]\big), \quad \tau \in [t, \min\{t + h_0, \vartheta\}].$$

Hence, we have $\theta(\tau - t)/(\tau - t) \to 0$ as $\tau \to t+0$. It means that the mapping g is ci-differentiable at the point $(t, x_t[\cdot])$. Further, by uniqueness of ci-derivatives, we obtain the inclution $(t, x_t[\cdot]) \in \mathbb{G}_*$ and equality (12). ■

For differential game (1), (2), we define the Hamiltonian

$$H\big(t, w[\cdot], s\big) := \min_{u \in \mathbb{U}} \max_{v \in \mathbb{V}} \big(\langle f\big(t, w[\cdot], u, v\big), s\rangle + \chi\big(t, w[\cdot], u, v\big)\big), \quad \big(t, w[\cdot]\big) \in \mathbb{G}, \quad s \in \mathbb{R}^n. \quad (13)$$

and consider, for a functional $\varphi \colon \mathbb{G} \mapsto \mathbb{R}$, the HJ equation

$$\partial_t \varphi\big(t, w[\cdot]\big) + \langle \partial_t g\big(t, w[\cdot]\big), \nabla\varphi\big(t, w[\cdot]\big)\rangle + H\big(t, w[\cdot], \nabla\varphi\big(t, w[\cdot]\big)\big) = 0, \quad \big(t, w[\cdot]\big) \in \mathbb{G}_*, \quad (14)$$

with the terminal condition

$$\varphi\big(\vartheta, w[\cdot]\big) = \sigma\big(w[\cdot]\big), \quad w[\cdot] \in \mathrm{Lip}. \quad (15)$$

The main difference between equation (14) and the HJ equation for retarded functional-differential systems (see, e.g., Lukoyanov (2000, 2003)) is the presence of the term $\langle \partial_t g(t, w[\cdot]), \nabla\varphi(t, w[\cdot])\rangle$. Since this term is well-defined only for $(t, w[\cdot]) \in \mathbb{G}_*$, equation (14) is also considered only for $(t, w[\cdot]) \in \mathbb{G}_*$. Nevertheless, a solution φ of problem (14), (15) should be defined on the whole set $\mathbb{G}$.

4. OPTIMAL STRATEGIES

Let us consider a functional $\varphi \colon \mathbb{G} \mapsto \mathbb{R}$ satisfying the following smoothness conditions:

(φ.1) For any $\nu > 0$, there exists $\lambda_\varphi > 0$ such that

$$\big|\varphi\big(t, w[\cdot]\big) - \varphi\big(\tau, r[\cdot]\big)\big| \le \lambda_\varphi\Big(\big|t - \tau\big| + \big\|w[\cdot] - r[\cdot]\big\|_\infty\Big)$$

for any $(t, w[\cdot]), (\tau, r[\cdot]) \in [t_0, \vartheta] \times D_\nu$.

(φ.2) The functional φ is ci-differentiable on the set $\mathbb{G}_*$.

(φ.3) There exist instants $t_0 < t_1 < \ldots < t_l = \vartheta$ such that, for any $\eta \in (0, \Delta t)$, where $\Delta t = \min\{t_i - t_{i-1}, i = \overline{1, l}\}$, and any $\nu > 0$, the ci-gradient $\nabla\varphi(t, w[\cdot])$ is uniformly continuous on the set $(\cup_{i=1}^{l} [t_{i-1}, t_i - \eta] \times D_\nu) \cap \mathbb{G}_*$.

Lemma 2. Let a functional $\varphi \colon \mathbb{G} \mapsto \mathbb{R}$ satisfy (φ.1) and (φ.2). Then, for any $\nu > 0$, there exists $K_\nabla > 0$ such that

$$\big\|\nabla\varphi\big(t, w[\cdot]\big)\big\| \le K_\nabla, \quad \big(t, w[\cdot]\big) \in \big([t_0, \vartheta] \times D_\nu\big) \cap \mathbb{G}_*. \quad (16)$$

Proof. Let λ_φ be defined by $\nu_* = \nu + \nu(\vartheta - t_0)$ according to (φ.1). Put $K_\nabla = 2\lambda_\varphi$. Let $(t, w[\cdot]) \in ([t_0, \vartheta] \times D_\nu) \cap \mathbb{G}_*$. If $\nabla\varphi(t, w[\cdot]) = 0$, then inequality (16) is obviously valid. Let $\nabla\varphi(t, w[\cdot]) \neq 0$. Let functions $\tilde{x}[\cdot], \hat{x}[\cdot] \in X(t, w[\cdot])$ be defined for $\tau \in [t, \vartheta]$ by the following rule:

$$\tilde{x}[\tau] = w[0], \quad \hat{x}[\tau] = w[0] + \nu(\tau - t)\frac{\nabla\varphi(t, w[\cdot])}{\|\nabla\varphi(t, w[\cdot])\|}.$$

Let $\tau \in [t, \vartheta]$. Due to (φ.2), we have

$$\varphi\big(\tau, \hat{x}_\tau[\cdot]\big) - \varphi\big(\tau, \tilde{x}_\tau[\cdot]\big) = \Big(\varphi\big(\tau, \hat{x}_\tau[\cdot]\big) - \varphi\big(t, w[\cdot]\big)\Big) - \Big(\varphi\big(\tau, \tilde{x}_\tau[\cdot]\big) - \varphi\big(t, w[\cdot]\big)\Big) = \nu\big\|\nabla\varphi\big(t, w[\cdot]\big)\big\|(\tau - t) + o_1(\tau - t) - o_2(\tau - t). \quad (17)$$

Moreover, since the inclusions $\tilde{x}_\tau[\cdot], \hat{x}_\tau[\cdot] \in D_{\nu_*}$ are valid, due to (φ.1), we obtain

$$\big|\varphi\big(\tau, \tilde{x}_\tau[\cdot]\big) - \varphi\big(\tau, \hat{x}_\tau[\cdot]\big)\big| \le \lambda_\varphi\big(\big\|\tilde{x}_\tau[\cdot] - w[\cdot]\big\|_\infty + \big\|\hat{x}_\tau[\cdot] - w[\cdot]\big\|_\infty\big) \le 2\lambda_\varphi \nu(\tau - t). \quad (18)$$

Inequality (16) follows from the relations (17) and (18), if we divide them by $(\tau - t)$ and tend τ to t. ■

In order to define players' control strategies by the extremal shift method in the direction of $\nabla\varphi(t, w[\cdot])$, this ci-gradient should be defined not only for $(t, w[\cdot]) \in \mathbb{G}_*$ (see (φ.2)), but for $(t, w[\cdot]) \in [t_0, \vartheta) \times \mathrm{Lip}$. Therefore, let us define a mapping $\Phi \colon [t_0, \vartheta) \times \mathrm{Lip} \mapsto \mathbb{R}^n$ by the following rule: if $(t, w[\cdot]) \in \mathbb{G}_*$, then $\Phi(t, w[\cdot]) := \nabla\varphi(t, w[\cdot])$; if $(t, w[\cdot]) \in ([t_0, \vartheta) \times \mathrm{Lip}) \setminus \mathbb{G}_*$, then

$$\Phi\big(t, w[\cdot]\big) := \lim_{k \to \infty} \nabla\varphi\big(t_k, x_{t_k}[\cdot]\big), \quad (19)$$

where $x[\cdot] \in X(t, w[\cdot])$ and the sequence t_k, $k = 1, 2, \ldots$, is such that $(t_k, x_{t_k}[\cdot]) \in \mathbb{G}_*$ for any $k = 1, 2, \ldots$; $t_k \to t + 0$

when $k \to \infty$; and the sequence $\nabla\varphi(t_k, x_{t_k}[\cdot])$ has a limit when $k \to \infty$. The existence of such a sequence follows from Lemmas 1 and 2. Moreover, using $(\varphi.3)$, one can show that the value $\Phi(t, w[\cdot])$ does not depend on the choice of the function $x[\cdot] \in X(t, w[\cdot])$ and the sequence t_k, $k = 1, 2, \ldots$, satisfying the conditions mentioned above.

The following properties of Φ can be proved:

$(\Phi.1)$ For any $\nu > 0$, for the number K_∇ from Lemma 2, we have

$$\big\|\Phi\big(t, w[\cdot]\big)\big\| \leq K_\nabla, \quad \big(t, w[\cdot]\big) \in [t_0, \vartheta) \times D_\nu.$$

$(\Phi.2)$ Let t_i, $i = \overline{0, l}$, and Δt be taken from $(\varphi.3)$. Then, for any $\eta \in (0, \Delta t)$ and any $\nu > 0$, the mapping Φ is uniformly continuous on $\cup_{i=1}^{l} [t_{i-1}, t_i - \eta] \times D_\nu$.

Let us consider the following players' control strategies:

$$\begin{aligned} U^\circ\big(t, w[\cdot]\big) &\in \operatorname*{argmin}_{u\in\mathbb{U}} \max_{v\in\mathbb{V}} s\big(t, w[\cdot], u, v\big), \\ V^\circ\big(t, w[\cdot]\big) &\in \operatorname*{argmax}_{v\in\mathbb{V}} \min_{u\in\mathbb{U}} s\big(t, w[\cdot], u, v\big), \end{aligned} \tag{20}$$

where $\big(t, w[\cdot]\big) \in [t_0, \vartheta) \times \mathrm{Lip}$ and

$$s\big(t, w[\cdot], u, v\big) = \langle f\big(t, w[\cdot], u, v\big), \Phi\big(t, w[\cdot]\big)\rangle + \chi\big(t, w[\cdot], u, v\big).$$

Theorem 1. Let a continuous functional $\varphi\colon \mathbb{G} \mapsto \mathbb{R}$ satisfy HJ equation (14) with terminal condition (15) and the smoothness conditions $(\varphi.1)$–$(\varphi.3)$. Then the control strategies U° and V° defined by (20) are optimal, and φ is the value functional of differential game (1), (2).

Proof. The proof is carried out by the scheme from Lukoyanov (2003). Let $(t_*, x_*[\cdot]) \in \mathbb{G}$. If $t_* = \vartheta$, then the validity of the theorem follows from $(\rho.1)$ and (15). Let $t_* < \vartheta$. In accordance with (7), (9) and (10), to prove the theorem, it is sufficient to prove the inequalities

$$\rho_u\big(t_*, x_*[\cdot]; U^\circ\big) \leq \varphi\big(t_*, x_*[\cdot]\big) \leq \rho_v\big(t_*, x_*[\cdot]; V^\circ\big). \tag{21}$$

Let us prove the first one. Due to (6), we should show that, for any $\zeta > 0$, there exists $\delta > 0$ such that the following statement is valid. Let Δ_δ be a partition (4). Then, for the control process realization $\{x[\cdot], u[\cdot], v[\cdot]\}$ generated by the control law of the first player $\{U^\circ, \Delta_\delta\}$ and an admissible control realization of the second player $v[\cdot]$, the value $\gamma = \gamma\big(t_*, x_*[\cdot]; U^\circ, \Delta_\delta; v[\cdot]\big)$ of quality index (2) satisfies the inequality

$$\begin{aligned} \gamma = \sigma\big(x_\vartheta[\cdot]\big) + \int_{t_*}^{\vartheta} \chi\big(\xi, x_\xi[\cdot], u[\xi], v[\xi]\big)\,\mathrm{d}\xi \\ \leq \varphi\big(t_*, x_*[\cdot]\big) + \zeta. \end{aligned} \tag{22}$$

Let X_* be the set of all functions $x[\cdot] \in X(t_*, x_*[\cdot])$ satisfying the following inequality for almost all $t \in [t_*, \vartheta]$:

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}t}\Big(x[t] - g\big(t, x_t[\cdot]\big)\Big)\Big\| \leq \alpha_f\big(1 + \|x_t[\cdot]\|_\infty\big). \tag{23}$$

Here α_f is taken from (f). One can show that, due to (g), there exist a number $\nu > 0$ such that

$$x_t[\cdot] \in D_\nu \tag{24}$$

for any $t \in [t_*, \vartheta]$ and $x[\cdot] \in X_*$, where D_ν is defined by (3).

Let t_i, $i = \overline{0, l}$, and Δt be taken from $(\varphi.3)$. Denote $i_* = \min\{i = \overline{1, l}\colon t_* < t_i\}$. By the continuity of χ and φ, due to (24), one can choose $\eta > 0$ such that

$$\Big|\varphi\big(t, x_t[\cdot]\big) + \int_\tau^t \chi\big(\xi, x_\xi[\cdot], u[\xi], v[\xi]\big)\mathrm{d}\xi - \varphi\big(\tau, x_\tau[\cdot]\big)\Big| \leq \zeta/(4l),$$

for any $t, \tau \in [t_i - \eta, t_i + \eta] \cap [t_*, \vartheta]$, $i = \overline{i_*, l}$, $x[\cdot] \in X_*$, and any admissible realizations $u[\cdot]$, $v[\cdot]$.

Denote $\zeta_* = \zeta/(4(\vartheta - t_0))$. By the continuity of f and χ, taking $(\Phi.1)$, $(\Phi.2)$ and (24) into account, for the functions H and s (see (13) and (20)), one can choose $\delta \in (0, \eta)$ such that, for any $x[\cdot] \in X_*$, $i = \overline{i_*, l}$, $t, \tau \in [t_{i-1}, t_i] \cap [t_*, \vartheta]$, $u \in \mathbb{U}$ and $v \in \mathbb{V}$, if $|t - \tau| \leq \delta$, then

$$\begin{aligned} \big\|s\big(t, x_t[\cdot], u, v\big) - s\big(\tau, x_\tau[\cdot], u, v\big)\big\| \leq \zeta_*, \\ \big\|H\big(t, x_t[\cdot], \Phi\big(t, x_t[\cdot]\big)\big) - H\big(\tau, x_\tau[\cdot], \Phi\big(\tau, x_\tau[\cdot]\big)\big)\big\| \leq \zeta_*. \end{aligned} \tag{25}$$

Let us show that this δ satisfies the statement above. Let Δ_δ be a partition (4) and the control process realization $\{x[\cdot], u[\cdot], v[\cdot]\}$ be generated by the control law $\{U^\circ, \Delta_\delta\}$ and an admissible realization $v[\cdot]$. Note that, according to (f) and (23), we have $x[\cdot] \in X_*$. Denote

$$\omega[t] := \varphi\big(t, x_t[\cdot]\big) + \int_{t_*}^{t} \chi\big(\xi, x_\xi[\cdot], u[\xi], v[\xi]\big)\,\mathrm{d}\xi, \quad t \in [t_*, \vartheta].$$

So, we have $\omega[t_*] = \varphi\big(t_*, x_*[\cdot]\big)$ and $\omega[\vartheta] = \gamma$. Due to the choice of δ, for every $i = \overline{i_*, l}$, there exists $k_i \in \mathbb{N}$ such that $\tau_{k_i} \in \Delta_\delta$ and $t_i \leq \tau_{k_i} \leq t_i + \eta$. Set $k_{i_*-1} = 1$. In particular, we have $\tau_{k_{i_*-1}} = t_*$ and $\tau_{k_l} = \vartheta$. According to the choice of η, we obtain

$$\begin{aligned} \omega[\vartheta] = \omega[t_*] + \sum_{i=i_*}^{l} \big(\omega[t_i - \eta] - \omega[\tau_{k_{i-1}}]\big) \\ + \sum_{i=i_*}^{l} \big(\omega[\tau_{k_i}] - \omega[t_i - \eta]\big) \leq \sum_{i=i_*}^{l} \int_{\tau_{k_{i-1}}}^{t_i - \eta} \frac{\mathrm{d}\omega[\xi]}{\mathrm{d}\xi}\,\mathrm{d}\xi + \zeta/2. \end{aligned}$$

Hence, to prove inequality (22), it is sufficient to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega[t] \leq 2\zeta_* \text{ for almost all } t \in [\tau_{k_{i-1}}, t_i - \eta],\ i = \overline{i_*, l}. \tag{26}$$

Let $i = \overline{i_*, l}$ and T_i be the set of all $t \in (\tau_{k_{i-1}}, t_i - \eta)$ such that $(t, x_t[\cdot]) \in \mathbb{G}_*$ and the derivatives $\mathrm{d}\varphi(t, x_t[\cdot])/\mathrm{d}t$ and $\mathrm{d}x[t]/\mathrm{d}t$ exist. By Lemma 1 and $(\varphi.1)$, the measure of the set $[\tau_{k_{i-1}}, t_i - \eta] \backslash T_i$ equals zero. For any $t \in T_i$, using $(\varphi.2)$ and (14), where, according to Lemma 1 and the definition of Φ, we substitute $\mathrm{d}g(t, x_t[\cdot])/\mathrm{d}t$ and $\Phi(t, x_t[\cdot])$ instead of $\partial_t g(t, x_t[\cdot])$ and $\nabla\varphi(t, x_t[\cdot])$, respectively, we deduce

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\varphi\big(t, x_t[\cdot]\big)\Big) &= \lim_{\tau \to t+0} \frac{\varphi(\tau, x_\tau[\cdot]) - \varphi(t, x_t[\cdot])}{\tau - t} \\ &= \partial_t\varphi\big(t, x_t[\cdot]\big) + \langle \frac{\mathrm{d}x[t]}{\mathrm{d}t}, \nabla\varphi\big(t, x_t[\cdot]\big)\rangle \\ &= \langle \frac{\mathrm{d}}{\mathrm{d}t}\Big(x[t] - g\big(t, x_t[\cdot]\big)\Big), \Phi\big(t, x_t[\cdot]\big)\rangle - H\big(t, x_t[\cdot], \Phi\big(t, x_t[\cdot]\big)\big). \end{aligned}$$

Thus, for almost all $t \in [\tau_{k_{i-1}}, t_i - \eta]$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\omega[t] &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\varphi\big(t, x_t[\cdot]\big)\Big) + \chi\big(t, x_t[\cdot], u[t], v[t]\big) \\ &= s\big(t, x_t[\cdot], u[t], v[t]\big) - H\big(t, x_t[\cdot], \Phi\big(t, x_t[\cdot]\big)\big). \end{aligned} \tag{27}$$

According to definition (20) of the strategy U° and taking into account definition (13) of H, for any $j = \overline{k_{i-1}, k_i}$ and $t \in [\tau_j, \tau_{j+1}] \cap [\tau_{k_{i-1}}, t_i - \eta]$, we obtain

$$\begin{aligned} s\big(\tau_j, x_{\tau_j}[\cdot], u[t], v[t]\big) &= s\big(\tau_j, x_{\tau_j}[\cdot], U^\circ\big(\tau_j, x_{\tau_j}[\cdot]\big), v[t]\big) \\ &\leq \max_{v\in\mathbb{V}} s\big(\tau_j, x_{\tau_j}[\cdot], U^\circ\big(\tau_j, x_{\tau_j}[\cdot]\big), v\big) \\ &= H\big(\tau_j, x_{\tau_j}[\cdot], \Phi\big(\tau_j, x_{\tau_j}[\cdot]\big)\big). \end{aligned} \tag{28}$$

From (25), (27) and (28), we conclude (26). Thus, the first inequality in (21) is proved. Due to $(f, \chi.2)$, the second inequality in (21) can be proved in a similar way. ∎

5. CI-DIFFERENTIABILITY PROPERTIES OF THE VALUE FUNCTIONAL

In order to prove the next theorem, following the ideas from Subbotin (1995) (see also Lukoyanov (2000)), we define auxiliary characteristic complexes and establish stability properties of the value functional of differential game (1), (2) with respect to these complexes.

Let us define the multivalued mappings

$$E_v\big(t, w[\cdot], v\big) := \mathrm{co}\,\big\{e\big(t, w[\cdot], u, v\big) \in \mathbb{R}^{n+1} \colon u \in \mathbb{U}\big\},$$
$$E_u\big(t, w[\cdot], u\big) := \mathrm{co}\,\big\{e\big(t, w[\cdot], u, v\big) \in \mathbb{R}^{n+1} \colon v \in \mathbb{V}\big\},$$
$$\big(t, w[\cdot]\big) \in \mathbb{G}, \quad u \in \mathbb{U}, \quad v \in \mathbb{V},$$

where $e(t, w[\cdot], u, v) := (f(t, w[\cdot], u, v), \chi(t, w[\cdot], u, v))$ and the symbol "co" denotes the convex hull of the corresponding set. Using the properties of the functions f and χ, by analogy with (Subbotin, 1995, p. 126), one can show that these mappings E_v and E_u have the following properties:

$(E.1)$ For any $(t, w[\cdot]) \in \mathbb{G}$, $u \in \mathbb{U}$ and $v \in \mathbb{V}$, the sets $E_v(t, w[\cdot], v)$ and $E_u(t, w[\cdot], u)$ are convex and compact.

$(E.2)$ For any $u \in \mathbb{U}$ and $v \in \mathbb{V}$, the mappings $E_v(t, w[\cdot], v)$ and $E_u(t, w[\cdot], u)$ are upper semicontinuous in $(t, w[\cdot]) \in \mathbb{G}$.

$(E.3)$ For any $(t, w[\cdot]) \in \mathbb{G}$, $u \in \mathbb{U}$, $v \in \mathbb{V}$ and $(f, \chi) \in E_v\big(t, w[\cdot], v\big) \cup E_u\big(t, w[\cdot], u\big)$, we have

$$\|f\| \leq \alpha_f\big(1 + \|w[\cdot]\|_\infty\big),$$

where α_f is taken from (f).

$(E.4)$ For any $(t, w[\cdot]) \in \mathbb{G}$ and $s \in \mathbb{R}^n$, we have

$$\max_{v \in \mathbb{V}} \min_{(f,\chi) \in E_v(t,w[\cdot],v)} \big(\langle s, f\rangle + \chi\big) = H\big(t, w[\cdot], s\big),$$
$$\min_{u \in \mathbb{U}} \max_{(f,\chi) \in E_u(t,w[\cdot],u)} \big(\langle s, f\rangle + \chi\big) = H\big(t, w[\cdot], s\big).$$

For any $t \in [t_0, \vartheta]$, let us consider the set

$$Z(t) := \big\{z[\cdot] \in \mathrm{Lip}\big([t-h, \vartheta], \mathbb{R}\big) \colon z[\xi] = 0,\ \xi \in [t-h, t]\big\}.$$

For $(t_*, x_*[\cdot]) \in \mathbb{G}$ and $v \in \mathbb{V}$, we denote by $CH_v(t_*, x_*[\cdot], v)$ the set of all pairs $(x[\cdot], z[\cdot]) \in X(t_*, x_*[\cdot]) \times Z(t_*)$ satisfying the neutral-type functional-differential inclusion

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big(x[t], z[t]\big) - \big(g\big(t, x_t[\cdot]\big), 0\big)\Big) \in E_v\big(t, x_t[\cdot], v\big) \qquad (29)$$

for almost all $t \in [t_*, \vartheta]$. Similarly, for $(t_*, x_*[\cdot]) \in \mathbb{G}$ and $u \in \mathbb{U}$, we denote by $CH_u(t_*, x_*[\cdot], u)$ the set of all pairs $x[\cdot] \in X(t_*, x_*[\cdot]) \times Z(t_*)$ which satisfy the inclusion

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big(x[t], z[t]\big) - \big(g\big(t, x_t[\cdot]\big), 0\big)\Big) \in E_u\big(t, x_t[\cdot], u\big) \qquad (30)$$

for almost all $t \in [t_*, \vartheta]$. Inclusions (29) and (30) are called the upper and the lower characteristic complexes.

From the properties of the function g and $(E.1)$–$(E.3)$, it follows that the sets $CH_v(t_*, x_*[\cdot], v)$ and $CH_u(t_*, x_*[\cdot], u)$ are non-empty compact subsets of $\mathrm{Lip}([t_* - h, \vartheta], \mathbb{R}^n \times \mathbb{R})$. Taking this fact into account, one can show that, due to $(\rho.2)$ and $(\rho.3)$, the value functional ρ° satisfies the following stability properties:

$(\rho.4)$ For any $(t_*, x_*[\cdot]) \in \mathbb{G}$ and $v \in \mathbb{V}$, there exists $(x[\cdot], z[\cdot]) \in CH_v(t_*, x_*[\cdot], v)$ such that

$$\rho^\circ\big(t, x_t[\cdot]\big) + z[t] \leq \rho^\circ\big(t_*, x_*[\cdot]\big), \quad t \in [t_*, \vartheta]. \qquad (31)$$

$(\rho.5)$ For any $(t_*, x_*[\cdot]) \in \mathbb{G}$ and $u \in \mathbb{U}$, there exists $(x[\cdot], z[\cdot]) \in CH_u(t_*, x_*[\cdot], u)$ such that

$$\rho^\circ\big(t, x_t[\cdot]\big) + z[t] \geq \rho^\circ\big(t_*, x_*[\cdot]\big), \quad t \in [t_*, \vartheta].$$

Theorem 2. Let the value functional $\rho^\circ \colon \mathbb{G} \mapsto \mathbb{R}$ of differential game (1), (2) be ci-differentiable at a point $(t_*, x_*[\cdot]) \in \mathbb{G}_*$. Then it satisfies HJ equation (14) at this point.

Proof. Let $\partial_t\rho^\circ_* := \partial_t\rho^\circ\big(t_*, x_*[\cdot]\big)$, $\nabla\rho^\circ_* := \nabla\rho^\circ\big(t_*, x_*[\cdot]\big)$ and $\varepsilon > 0$ be fixed. By $(E.4)$, there exists $v_* \in \mathbb{V}$ such that

$$\min_{(f,\chi) \in E_*} \big(\langle f, \nabla\rho^\circ_*\rangle + \chi\big) \geq H\big(t_*, x_*[\cdot], \nabla\rho^\circ_*\big) - \varepsilon/2, \qquad (32)$$

where $E_* := E_v(t_*, x_*, v_*)$. Due to $(\rho.4)$, one can choose a pair $(x[\cdot], z[\cdot]) \in CH_v(t_*, w_*[\cdot], v)$ such that inequality (31) is valid. Taking into account ci-differentiability of ρ° at $(t_*, x_*[\cdot])$, for any $t \in [t_*, \vartheta]$, we deduce

$$\begin{aligned} 0 &\geq \rho^\circ\big(t, x_t[\cdot]\big) - \rho^\circ\big(t_*, x_*[\cdot]\big) + z[t] \\ &= (t - t_*)\partial_t\rho^\circ_* + \langle x[t] - x[t_*], \nabla\rho^\circ_*\rangle + z[t] + o_1(t - t_*). \end{aligned} \qquad (33)$$

Since $(x[\cdot], z[\cdot]) \in CH_v(t_*, w_*[\cdot], v)$, there exist measurable functions $\tilde{f}[\cdot] \colon [t_*, \vartheta] \mapsto \mathbb{R}^n$ and $\tilde{\chi}[\cdot] \colon [t_*, \vartheta] \mapsto \mathbb{R}$ such that

$$\big(\tilde{f}[t], \tilde{\chi}[t]\big) \in E_v\big(t, x_t[\cdot], v_*\big), \quad t \in [t_*, \vartheta], \qquad (34)$$

and

$$\begin{aligned} x[t] &= g\big(t, x_t[\cdot]\big) + x[t_*] - g\big(t_*, x_*[\cdot]\big) + \int_{t_*}^{t} \tilde{f}[\xi]\,\mathrm{d}\xi, \\ z[t] &= \int_{t_*}^{t} \tilde{\chi}[\xi]\,\mathrm{d}\xi, \quad t \in [t_*, \vartheta]. \end{aligned} \qquad (35)$$

Since $(t_*, x_*[\cdot]) \in \mathbb{G}_*$, then $\partial_t g_* := \partial_t g(t_*, x_*[\cdot])$ exists, and, for any $t \in [t_*, \vartheta]$, we have

$$g\big(t, x_t[\cdot]\big) - g\big(t_*, x_*[\cdot]\big) = (t - t_*)\partial_t g_* + o_2(t - t_*). \qquad (36)$$

Denote

$$\theta_*(t) := o_1(t - t_*)/(t - t_*) + \langle o_2(t - t_*), \nabla\rho^\circ_*\rangle/(t - t_*).$$

Dividing (33) by $(t - t_*)$ and using (35), (36), we obtain

$$\begin{aligned} 0 \geq \partial_t\rho^\circ_* + \langle \partial_t g_*, \nabla\rho^\circ_*\rangle + \Big\langle \frac{1}{t - t_*}\int_{t_*}^{t} \tilde{f}[\xi]\,\mathrm{d}\xi, \nabla\rho^\circ_* \Big\rangle \\ + \frac{1}{t - t_*}\int_{t_*}^{t} \tilde{\chi}[\xi]\,\mathrm{d}\xi + \theta_*(t), \quad t \in (t_*, \vartheta]. \end{aligned} \qquad (37)$$

Due to $(E.2)$ and (34), for $\varepsilon_1 = \varepsilon/(4(\|\nabla\rho^\circ_*\| + 1))$, there exists $\nu > 0$ such that

$$\big(\tilde{f}[\xi], \tilde{\chi}[\xi]\big) \in \big[E_*\big]^{\varepsilon_1}, \quad \xi \in [t_*, t_* + \nu].$$

Hence, using Lemma 12 from (Filippov, 1988, p. 63) and $(E.1)$, we obtain that, for any $t \in [t_*, t_* + \nu]$, there exists a pair $(f_*[t], \chi_*[t]) \in E_*$ satisfying the inequalities

$$\Big\| \frac{1}{t - t_*}\int_{t_*}^{t} \tilde{f}[\xi]\,\mathrm{d}\xi - f_*[t] \Big\| \leq \varepsilon_1, \ \Big| \frac{1}{t - t_*}\int_{t_*}^{t} \tilde{\chi}[\xi]\,\mathrm{d}\xi - \chi_*[t] \Big| \leq \varepsilon_1.$$

Using these inequalities and (32), from (37) we obtain

$$\begin{aligned} \varepsilon &\geq \varepsilon/2 + \partial_t\rho^\circ_* + \langle \partial_t g_*, \nabla\rho^\circ_*\rangle + \langle f_*[t], \nabla\rho^\circ_*\rangle + \chi_*[t] + \theta_*(t) \\ &\geq \partial_t\rho^\circ_* + \langle \partial_t g_*, \nabla\rho^\circ_*\rangle + H\big(t_*, x_*[\cdot], \nabla\rho^\circ_*\big) + \theta_*(t). \end{aligned}$$

Letting t to t_* and, after that, ε to 0, we conclude

$$\partial_t\rho^\circ_* + \langle \partial_t g_*, \nabla\rho^\circ_*\rangle + H\big(t_*, x_*[\cdot], \nabla\rho^\circ_*\big) \leq 0.$$

The inequality

$$\partial_t\rho^\circ_* + \langle \partial_t g_*, \nabla\rho^\circ_*\rangle + H\big(t_*, x_*[\cdot], \nabla\rho^\circ_*\big) \geq 0$$

can be proved in a similar way on the basis of $(\rho.5)$. ∎

6. EXAMPLES

Consider a two-person zero-sum differential game for the dynamical system described by the neutral-type functional-differential equation in Hale's form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(x[t]-x[t-1]\big)=u[t]-v[t],\qquad t\in[0,3],\quad x[t]\in\mathbb{R},\quad u[t],v[t]\in[-1,1],\tag{38}$$

and the quality index

$$\gamma_1=\sigma\big(x_{\vartheta=3}[\cdot]\big)=\big|x[3]\big|^2.\tag{39}$$

It is known (see, e.g., Gomoyunov and Lukoyanov (2018)) that this differential game has the value ρ_1°.

Note that, for any control realization $u[\cdot]$ of the first player, there exists the control realization $v[\cdot]\equiv u[\cdot]$ of the second player, and vice versa. Therefore, the value $\rho_1^\circ\big(t,w[\cdot]\big)$ equals to the value of quality index (39) that corresponds to the case when $v[\cdot]\equiv u[\cdot]$. Hence, we obtain

$$\rho_1^\circ\big(t,w[\cdot]\big)=\big|\kappa\big(t,w[\cdot]\big)\big|^2,\qquad \kappa\big(t,w[\cdot]\big)=w[i-t]+(3-i)w[0]-(3-i)w[-1],\qquad \big(t,w[\cdot]\big)\in[i,i+1]\times\mathrm{Lip},\quad i=\overline{0,2}.\tag{40}$$

According to (11) and (13), we have

$$\mathbb{G}_*=\Big\{\big(t,w[\cdot]\big)\in\mathbb{G}\colon\exists\frac{\mathrm{d}^+}{\mathrm{d}t}w[-1]\Big\},\ H\big(t,w[\cdot],s\big)\equiv0,\tag{41}$$

where $\mathrm{d}^+w[-1]/\,\mathrm{d}t$ is the right-hand side derivative of the function $w[\cdot]$ at the point $\xi=-1$. One can show that the value functional ρ_1° has ci-derivatives

$$\partial_t\rho_1^\circ\big(t,w[\cdot]\big)=-2(3-i)\kappa\big(t,w[\cdot]\big)\,\mathrm{d}^+w[-1]/\,\mathrm{d}t,$$
$$\nabla\rho_1^\circ\big(t,w[\cdot]\big)=2(3-i)\kappa\big(t,w[\cdot]\big),$$
$$\big(t,w[\cdot]\big)\in\big([i,i+1)\times\mathrm{Lip}\big)\cap\mathbb{G}_*,\quad i=\overline{0,2},$$

and satisfies HJ equation (14), terminal condition (15) and the smoothness conditions (φ.1)–(φ.3). The corresponding functional Φ (see (19)) is defined by $\Phi(t,w[\cdot])=2(3-i)\kappa(t,w[\cdot])$. Thus, by Theorem 1, players' optimal strategies U° and V° can be constructed for $(t,w[\cdot])\in\mathbb{G}$, $\kappa(t,w[\cdot])\neq0$, by the formulas

$$U^\circ\big(t,w[\cdot]\big)=-\frac{\kappa(t,w[\cdot])}{\|\kappa(t,w[\cdot])\|},\quad V^\circ\big(t,w[\cdot]\big)=\frac{\kappa(t,w[\cdot])}{\|\kappa(t,w[\cdot])\|},$$

and, for $(t,w[\cdot])\in\mathbb{G}$, $\kappa(t,w[\cdot])=0$, they can be defined, for instance, by $U^\circ(t,w[\cdot])=V^\circ(t,w[\cdot])=0$.

As the second example, consider the differential game for the same dynamical system (38) and the quality index

$$\gamma_2=\sigma\big(x_{\vartheta=3}[\cdot]\big)=\big|x[3]\big|.$$

The value functional of this game is

$$\rho_2^\circ\big(t,w[\cdot]\big)=\big|\kappa\big(t,w[\cdot]\big)\big|,$$

where κ is taken from (40). Let $\mathbb{G}_*=\mathbb{G}_0\cup\mathbb{G}_-\cup\mathbb{G}_+$, where

$$\mathbb{G}_0:=\big\{\big(t,w[\cdot]\big)\in\mathbb{G}_*\colon\kappa\big(t,w[\cdot]\big)=0\big\},$$
$$\mathbb{G}_+:=\big\{\big(t,w[\cdot]\big)\in\mathbb{G}_*\colon\kappa\big(t,w[\cdot]\big)>0\big\},$$
$$\mathbb{G}_-:=\big\{\big(t,w[\cdot]\big)\in\mathbb{G}_*\colon\kappa\big(t,w[\cdot]\big)<0\big\}.$$

Note that the value functional ρ_2° is not ci-differentiable on the set $\mathbb{G}_*$ (in particular, at the points of $\mathbb{G}_0$). However, on the sets $\mathbb{G}_+$ and $\mathbb{G}_-$, the functional ρ_2° has ci-derivatives

$$\partial_t\rho_2^\circ\big(t,w[\cdot]\big)=-(3-i)\,\mathrm{d}^+w[-1]/\mathrm{d}t,\ \nabla\rho_2^\circ\big(t,w[\cdot]\big)=3-i,\qquad \big(t,w[\cdot]\big)\in\big([i,i+1)\times\mathrm{Lip}\big)\cap\mathbb{G}_+,$$

$$\partial_t\rho_2^\circ\big(t,w[\cdot]\big)=(3-i)\,\mathrm{d}^+w[-1]/\mathrm{d}t,\ \nabla\rho_2^\circ\big(t,w[\cdot]\big)=-(3-i),\qquad \big(t,w[\cdot]\big)\in\big([i,i+1)\times\mathrm{Lip}\big)\cap\mathbb{G}_-,$$

and satisfies HJ equation (14). This fact illustrates the statement of Theorem 2.

7. CONCLUSION

In the paper, the relationship between differential game (1), (2) and HJBI equation (14) with terminal condition (15) is given. According to Theorem 1, if this equation has a sufficiently smooth solution, then it coincides with the value functional, and players' optimal control strategies are constructed by the extremal shift in the direction of the continued ci-gradient of this solution (20). However, under the considered conditions, problem (14), (15) may not have such a smooth solution (see the second example in Section 6). This leads to the necessity of considering an appropriate generalized solution of problem (14), (15). This is the topic of future research.

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