An Algebraic Approach for a Stability Analysis Methodology for Multiple Time-delay Systems

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Abstract:
This paper presents an improvement on Cluster Treatment of Characteristic Roots (CTCR), which is a well-known methodology for delay-dependent stability analysis of multiple time-delay systems (MTDS). We propose an algebraic approach to extract the stability switching hypersurfaces in spectral delay space, instead of a numerical procedure in CTCR with Extended Kronecker Sum (EKS) operation. The proposed algebraic approach is based on an efficient zero location test, and the deployment of this test to an auxiliary characteristic polynomial whose unique properties have recently been revealed. The achieved improvement is demonstrated by applying the new CTCR procedure to a system with three delays.

Keywords: Time-delay systems, stability, Cluster Treatment of Characteristic Roots, Kronecker sum, Bistritz Tabulation, self-inversive polynomials

1. INTRODUCTION

In this paper, we study the stability of LTI-MTDS of retarded type, with the general state space representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \textstyle\sum_{j=1}^{p} \mathbf{B}_j \mathbf{x}(t-\tau_j), \qquad (1)$$

where the state vector is $\mathbf{x} \in \mathbb{R}^n$, the system matrices are $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B}_j \in \mathbb{R}^{n\times n}$, and $\tau_j \geq 0$ are constant time-delays, which are rationally independent from each other, i.e. incommensurate delays.

The stability analysis of the system (1) was declared as an NP-hard problem by Toker and Ozbay [1996], and have been widely studied via different approaches in time and frequency domains. The time-domain methods are based on Lyapunov-Razumikhin functions or Lyapunov-Krasovskii functionals (Fridman [2014]), where the problem is solved usually by the feasibility of linear matrix inequalities (LMIs). The frequency domain approaches mainly focus on the characteristic equation of (1) given by

$$CE(s,\boldsymbol{\tau}) = \det\left[s\mathbf{I} - \mathbf{A} - \textstyle\sum_{j=1}^{p} \mathbf{B}_j e^{-\tau_j s}\right] = 0, \qquad (2)$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_p)$ denotes the delay vector. The general motivation is to investigate the imaginary axis crossing of the roots of (2), which are infinitely many. CTCR is such a frequency domain stability paradigm, which treats the determination of characteristic roots on the imaginary axis ($s = \pm\omega i$), and then examines the (in)stability switchings of these roots in the complex plane.

CTCR paradigm has been proposed for single-delay systems in Olgac and Sipahi [2002] and then extended to multiple-delay case by Sipahi and Olgac [2005, 2006]. A computational advantage for the method has been provided by Ergenc et al. [2007], which utilizes the Extended Kronecker Sum (EKS) to determine the characteristic roots of (2) on the imaginary axis. It is worth here to refer Louisell [2001], Jarlebring [2009] that also utilize Kronecker sum to determine crossing frequencies and critical delay values. For the system (1) with $p > 3$, a frequency sweeping methodology and the resultant theory were utilized in Sipahi and Delice [2011] and in Gao and Olgac [2016] to improve the numerical procedure in CTCR method for extracting the 2-D cross-sections of stability switching hypersurfaces in delay space.

In this paper, we provide an algebraic approach to extract the stability switching hypersurfaces (a.k.a *building hypersurfaces*) in the spectral delay space ($\boldsymbol{\tau}\omega \in \mathbb{R}^p_+$) of the system (1). The proposed approach brings out a modified procedure for CTCR with EKS, which reduces the numerical calculations in the former procedure in Ergenc et al. [2007]. The new procedure is based on an efficient zero location test, the *Bistritz Tabulation*, and its application to an auxiliary characteristic polynomial which was revealed and investigated recently in Alikoc and Ergenc [2017]. Besides the employment of simple algebraic operations in the new method, the unique properties of the

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auxiliary polynomial are also utilized to reduce the effort for numerical calculations.

The paper is organized as follows: In section 2, preliminary definitions and statements of the study are given. The main result for the extraction of building hypersurfaces, and the modified CTCR method with EKS given in section 3. Section 4 contains an example case study. The paper is concluded in Section 5 with the results summary.

2. PRELIMINARIES

In this section, we represent an auxiliary characteristic polynomial obtained via EKS, its unique features, and its relation with the imaginary characteristic roots of (2). Then, a review of CTCR procedure is given. Finally, a zero location test named *Bistritz Tabulation*, which constitutes the basis of the improved methodology, is represented. We omit the proofs due to space limitation, an interested reader may see the cited studies for details.

2.1 Kronecker Sum and Imaginary Characteristic Roots

The following theorem was proposed in Ergenc et al. [2007], taking the advantage of "eigenvalue addition" feature of Kronecker summation operation.

Theorem 1. Define the Auxiliary Characteristic Equation (ACE) of the system given in (1), with $z_j = e^{-\tau_j s}$:

$$ACE(\mathbf{z}) = \det\left[\begin{array}{c}\mathbf{A}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{A}+\\ \sum_{j=1}^{p}(\mathbf{B}_j\otimes\mathbf{I}z_j+\mathbf{I}\otimes\mathbf{B}_j z_j^{-1})\end{array}\right]=0. \quad (3)$$

Then, for the system (1), the following findings are equivalent:

- (i) A vector of p-dimensional unitary complex numbers $\mathbf{z}=\{z_j\}\in\mathbb{T}^p$, $|z_j|=1, \forall j=1,\ldots,p$ satisfies ACE given in (3).
- (ii) There exists at least one pair of imaginary characteristic roots, $\pm\omega i$, of (2).
- (iii) There exists a corresponding delay vector $\boldsymbol{\tau}\in\mathbb{R}_+^p$ which satisfies $CE(\pm\omega i,\boldsymbol{\tau})=0$.

Theorem 1 states necessarily and sufficiently that one pair of imaginary characteristic poles, $\pm\omega i$, of (2) for a certain value of set $\boldsymbol{\tau}$ correspond to a unitary root set, $\mathbf{z}\in\mathbb{T}^p$. Note that ACE is free of s terms. Let the complete unitary solution set of (3) be

$$\mathbf{Z}=\{\mathbf{z}\in\mathbb{T}^p \mid ACE(\mathbf{z})=0\}. \quad (4)$$

Then the set of imaginary crossing frequencies and the set of corresponding delays which cause stability switching can be given, respectively, as

$$\Omega=\{\omega\in\mathbb{R}\mid CE(s=\omega i,\mathbf{z})=0,\mathbf{z}\in\mathbf{Z}\}, \quad (5)$$

$$\wp=\left\{\boldsymbol{\tau}\in\mathbb{R}_+^p \mid \langle\boldsymbol{\tau},\omega,\mathbf{z}\rangle,\boldsymbol{\tau}=\frac{\overline{\arg}(\mathbf{z})+2k\boldsymbol{\pi}}{\omega},\omega\in\Omega,\mathbf{z}\in\mathbf{Z}\right\}, \quad k=0,1,\ldots \quad (6)$$

where $\overline{\arg}(\cdot)$ denote the elementwise argument operation and $\boldsymbol{\pi}=(\pi,\ldots,\pi)\in\mathbb{R}^p$. Also, $\langle\cdot,\cdot,\cdot\rangle$ notation implies a causal relation that the p members of the first argument result in the second argument as the imaginary root of CE, and the first and second arguments together result in a unitary root of ACE. The whole delay set (6) resulting imaginary characteristic roots can be obtained easily after finding the switching hypersurface for $k=0$ which is called *kernel hypersurface* ($\wp_0$). Note that the hypersurfaces generated from the *kernel* are called *offspring*. Together with the hypersurfaces in delay space, a perspective with a conditional mapping from delay space, $\boldsymbol{\tau}\in\mathbb{R}_+^p$, only for the points $\boldsymbol{\tau}\in\wp$, to a *spectral delay space*, $\boldsymbol{\tau}\omega\in\mathbb{R}_+^p$, generated via $\langle\boldsymbol{\tau},\omega\rangle$ correspondence was defined in Fazelinia et al. [2007]. With this perspective, the *building hypersurface* is defined where $\boldsymbol{\tau}\in\wp_0$ are mapped into a bounded space named the *building block*, which is a p-dimensional cube with edge length of 2π, i.e. $0\le\tau_j\omega\le 2\pi$, $\forall j$. Clearly, this building hypersurface is defined as

$$\boldsymbol{\Delta}=\left\{\boldsymbol{\tau}\omega\in[0,2\pi]^p \;\middle|\; \boldsymbol{\tau}\omega=\overline{\arg}(\mathbf{z}),\mathbf{z}\in\mathbf{Z}\right\} \quad (7)$$

considering (4)–(6) as also mentioned in Ergenc et al. [2007]. The notation $\boldsymbol{\tau}\omega\in[0,2\pi]^p$ means that $\boldsymbol{\tau}\omega$ is a vector of $\mathbb{R}_+^p$ whose entries belong to the interval $[0,2\pi]$.

Let us now focus on the properties of auxiliary characteristic function $-ACE$, which is a multinomial in terms of $\mathbf{z}$ where the degree of any z_k is smaller than or equal to n^2. We have proven recently that the unitary zero sets of (3) can be represented via a *self-inversive* polynomial in terms of any $z_k\in\mathbf{z}$ with complex coefficients $z_{j\neq k},\forall j$; see the following definition and lemma.

Definition 1. Consider the following polynomial with complex coefficients:

$$P_n(z)=\sum_{i=0}^{n} d_i z^i\,. \quad (8)$$

The reciprocal of $P_n(z)$ is

$$P_n^*(z)=\sum_{i=0}^{n}\bar{d}_{n-i}z^i=z^n\bar{P}_n(1/z), \quad (9)$$

where $\bar{d}$ denotes the complex conjugate of d. Then, the polynomial is called as *symmetric* or *self-inversive* if $P_n(z)=P_n^*(z)$.

Lemma 1. (Alikoc and Ergenc [2017]) Define the Auxiliary Characteristic Function in (3) as $ACE(\mathbf{z})=n(\mathbf{z})/d(\mathbf{z})$. Then, the roots of (3) are equal to the zeros of the "Auxiliary Characteristic Polynomial (ACP)",

$$ACP(\mathbf{z})=z_k^{\rho_k}ACE(\mathbf{z}) \quad (10)$$

where ρ_k is the maximal degree of z_k in $d(\mathbf{z})$, $\forall k\in(1,\ldots,p)$, and $ACP(\mathbf{z})$ is a self-inversive polynomial of even degree in terms of z_k with complex coefficients $(z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_p)$ for $z_j=e^{i\omega\tau_j}$.

Let us rewrite (10) as

$$ACP(z_k,\boldsymbol{\upsilon})=\sum_{l=0}^{m_k}b_l^{(k)}(\boldsymbol{\upsilon})\,z_k^l, \quad (11)$$

where $m_k\le 2n^2$ and $b_l^{(k)}(\boldsymbol{\upsilon})$ are complex coefficients in terms of $e^{i\upsilon_j}$, where $\upsilon_j:=\tau_j\omega$ for all $j\neq k$. Obviously, the set

$$\boldsymbol{\Upsilon}=\left\{\boldsymbol{\upsilon}\in[0,2\pi]^{p-1} \;\middle|\; ACP(z_k,\boldsymbol{\upsilon})=0,|z_k|=1\right\} \quad (12)$$

is a subset of building hypersurfaces $\boldsymbol{\Delta}$ in (7), i.e. $\boldsymbol{\Upsilon}\subset\boldsymbol{\Delta}$, and $\upsilon_k\in\boldsymbol{\tau}\omega$ is represented by an unitary complex variable z_k. By the fact that $b_l^{(k)}(\boldsymbol{\upsilon}),\forall l$, has a periodicity of 2π and ACP is self-inversive, the following lemma was also proposed.

Lemma 2. (Alikoc and Ergenc [2017]) Any set of $\boldsymbol{\upsilon}'\in\boldsymbol{\Upsilon}$ resulting a unitary zero z_k^* of (11) is symmetric to the point

$\boldsymbol{\upsilon}'' \in \boldsymbol{\Upsilon}$ which satisfies $ACP(\bar{z}_k^*, \boldsymbol{\upsilon}'') = 0$ where also $\bar{z}_k^*$ is a unitary zero, with respect to the point $\boldsymbol{\pi}(\pi, \pi, \ldots, \pi) \in \mathbb{R}^{p-1}$. That is,

$$\frac{\upsilon_j' + \upsilon_j''}{2} = \pi, \; j = 1, 2, \ldots, k-1, k+1, \ldots, p. \qquad (13)$$

Notice that Lemma 2 provides the investigation of unitary zero sets of ACP by sweeping one of the parameters υ_j in the range $[0, \pi]$ and the others in $[0, 2\pi]$ instead of sweeping all for $[0, 2\pi]$. This fact is one of the utilized advantages to reduce computational load of the former CTCR procedure.

Remark 1. Self-inversive polynomials have zeros which are all unitary and/or reciprocal; meaning conjugate pairs symmetrical to unit circle. We also take the advantage of this feature of ACP.

2.2 Cluster Treatment of Characteristic Roots (CTCR)

CTCR is based on two distinctive properties: first, the determination of the stability switching hypersurfaces ($\wp$) resulting imaginary characteristic roots from a *finite* number of (kernel) hypersurfaces – $\wp_0$, and secondly, the invariance property of *root tendency* (RT) on $\wp$. Besides the approach presented by Theorem 1, these hypersurfaces can also be obtained by different approaches, e.g. by Rekasius Substitution in Sipahi and Olgac [2005, 2006] or by building block concept in Fazelinia et al. [2007].

RT defines the crossing direction of the characteristic roots on the imaginary axis. This direction is invariant to delay changes on the switching hypersurfaces, which occurs as a direct result of root continuity argument. With this invariance property, the determination of the number of the unstable roots in delay space is provided systematically. RT for a purely imaginary characteristic root (ωi) with respect to one of the time-delays, τ_j, is given in (Sipahi and Olgac [2006]) as

$$\mathrm{RT} = \mathrm{RT}|_{s=\omega i}^{\tau_j} = \operatorname{sgn}\left[\Re\left(\left.\frac{\partial CE/\partial \tau_j}{\partial CE/\partial s}\right|_{s=\omega i}\right)\right], \forall j. \qquad (14)$$

The RT function represents the direction of the root at $s = \omega i$ crossing when only one of the delays varies. If RT $= +1$, the imaginary root crosses to the right half complex-plane. Otherwise (RT = -1), it crosses to the left half complex-plane. Then, for each region in delay space whose boundaries are created by $\wp$ in (6), the number of roots in left (or right) half complex plane, i.e. the number of unstable roots, can be determined by obtaining the RT values and counting the roots in each plane. Calling the number of unstable roots as NU, clearly the system is said to be stable for the delay values in a generated region by hypersurfaces if and only if NU=0.

The procedure for CTCR method with EKS approach in Ergenc et al. [2007] is summarized as follows:

(i) Obtain the ACE in (3) for the system (1).
(ii) Select any z_k, $k \in (1, \ldots, p)$, as the complex variable and substitute $z_j = e^{i\upsilon_j}$, $\forall j \neq k$, $\upsilon_j \in [0, 2\pi]$, into (3). Solving the roots of (3) for all $\boldsymbol{\upsilon} \in [0, 2\pi]^{p-1}$, form the corresponding z_k values with unity magnitude. This yields the complete unitary solution set (4) of ACE and the building hypersurfaces (7).
(iii) Compute the set of all imaginary crossing frequencies (5) by inserting each $\mathbf{z} \in \mathbb{T}^p$ found in step (ii) and $s = \omega i$ to CE in (2).
(iv) Calculate the switching hypersurfaces of delays described in (6) with the corresponding $\mathbf{z}$ and ω, found respectively in step (ii) and step (iii).
(v) Finally, determine the directions of stability switching on the boundaries of hypersurfaces and NU for the regions in delay space, by RT function given in (14). The regions with NU=0 are stable.

Apparently, step (ii) and (iii) in the above CTCR procedure constitute the part with the severest computational load. As the main contribution of this work, we propose a new procedure for CTCR to reduce the mentioned computational load, based on the features of the auxiliary characteristic polynomial ACP given by (10)-(11), and the utilized zero location test represented below.

2.3 Bistritz Tabulation Method

Bistritz Tabulation (BT) is a Routh-like tabular method to find the number of the zeros of polynomials with respect to unit circle, which provides computational efficiency and easier implementation for unknown parameters, compared with the alternative methods based on Schur-Cohn matrices and Jury-Marden tables (see Bistritz [1984]). Before outlining the framework, let us first provide the necessary definitions below, for the method.

Definition 2. $P_n(z)$ in (8) is called *normal* if $d_n \neq 0$. Otherwise it is called *abnormal*. In other words, being *normal* is the equivalence of the formal degree (n) and the exact degree of the polynomial.

Definition 3. The *deficiency parameter*, λ_k, is the difference between the formal (i.e. expected) degree and the exact degree of a polynomial $P_k(z)$ where k denotes the degree of the polynomial. $P_k(z)$ is normal if $\lambda_k = 0$ and abnormal if $\lambda_k > 0$.

BT determines the number of the zeros of a polynomial inside ($\mathbb{D}$), on ($\mathbb{T}$) and outside ($\mathbb{S}$) the unit circle. It is based on a three-term recursion of symmetric polynomials and the number of sign variations of these polynomials at $z = 1$. The method was extended for polynomials with complex coefficients in Bistritz [1986]. Moreover, the algorithm was improved to overcome one of the singularity types and a more compact form is given in Bistritz [2002] that we outline below.

For a complex coefficient ($d_i \in \mathbb{C}$) polynomial $P_n(z)$ defined in (8), such that $P_n(1) \neq 0 \in \mathbb{R}$ and $d_n \neq 0$, the *regular* recursion algorithm is as follows:

$$T_n(z) = \sum_{i=0}^{n} t_{ni} z^i = P_n(z) + P_n^*(z), \qquad (15)$$

$$T_{n-1}(z) = \sum_{i=0}^{n-1} t_{(n-1)i} z^i = \frac{P_n(z) - P_n^*(z)}{z - 1}, \qquad (16)$$

For $k = n-1, \ldots, 0$

$$\delta_{k+1} = \begin{cases} \dfrac{t_{(k+1)0}}{t_{k\lambda_k}}, & \text{if } T_k(z) \not\equiv 0 \\ 0, & \text{if } t_{(k+1)0} = 0 \\ \text{not required}, & \text{if } t_{(k+1)0} \neq 0 \;\; \& \;\; T_k(z) \equiv 0 \end{cases}$$

$$T_{k-1}(z) = z^{-1}\left[\left(\delta_{k+1}z^{-\lambda_k} + \bar{\delta}_{k+1}z^{\lambda_k+1}\right)T_k(z) - T_{k+1}(z)\right], \tag{17}$$

where $P_n^*(z)$ is the reciprocal of $P_n(z)$, given by (9).

The following theorem is for counting the number of zeros inside and outside the unit circle in regular (i.e. nonsingular) case.

Theorem 2. (Bistritz [2002]) Consider $P_n(z)$ with the assumptions $P_n(1) \neq 0 \in \mathbb{R}$ and $d_n \neq 0$. Assume that the procedure is regular. Then,

(i) the number of zeros in $\mathbb{D}$: $\alpha_n = n - \nu_n$,
(ii) the number of zeros in $\mathbb{S}$: $\gamma_n = \nu_n$

where $\nu_n = \mathrm{Var}\{\sigma_n, \sigma_{n-1}, \ldots, \sigma_0\}$ such that $\sigma_k := T_k(1)$ and Var{.} denotes number of sign variations.

The only singularity situation which interrupts the regular recursion occurs if and only if a normal polynomial $T_\eta(z)$ ($\lambda_\eta = 0$) is followed by an identically zero polynomial, i.e. $T_{\eta-1}(z) \equiv 0$, in the recursion algorithm given above. Actually, this situation arise when the polynomial has unit circle and/or reciprocal zero(s), which is precisely the case for the self-inversive polynomial ACP (recall Remark 1). Instead of handling the obvious singularity that is to be faced at the beginning of BT applied to ACP, the following theorem is utilized in the main results.

Theorem 3. (Sheil-Small [2002]) Let P is a self-inversive polynomial of degree n. Suppose that P has exactly β zeros on the unit circle $\mathbb{T}$ (multiplicity included) and exactly μ critical points in the closed unit disc $\overline{\mathbb{D}}$ (counted according to multiplicity). Then,

$$\beta = 2(\mu + 1) - n. \tag{18}$$

Remark 2. It is useful to mention the case where the examined polynomial has complex variable coefficients for application of Bistritz method. In this case, the term $(z-1)$ in the denominator of the polynomial obtained by recursion eq. (16) is not vanished. This issue is fixed in Bistritz [1996] with a simple modification in the algorithm by applying the recursion algorithm to $(z-1)P_n(z)$ yielding the sequence $\bar{\sigma}_k = \{0, T_n(1), T_{n-1}(1), \ldots, T_0(1)\}$ where the first element is always zero as a consequent of multiplication with $(z-1)$. Then, the zero location of $P_n(z)$ is examined same as in Theorem 2 with respect to the sign variation in the list extracting first "0" element, i.e. $\sigma_k = \{T_n(1), T_{n-1}(1), \ldots, T_0(1)\}$.

The modification described above will also be utilized to test the location of zeros of ACP for determining the set $\boldsymbol{\Upsilon}$ in (12), and correspondingly the $\boldsymbol{\Delta}$ in (7).

3. MAIN RESULTS

Based on the represented results in the previous section, we provide the following corollary.

Corollary 1. Consider $ACP(z_k, \boldsymbol{v})$ in (11) obtained by (3) and (10) for any z_k, $k \in (1, \ldots, p)$ for the system (1). Defining the polynomial,

$$D(z_k, \boldsymbol{v}) := (z_k - 1)\frac{\partial ACP(z_k, \boldsymbol{v})}{\partial z_k} \tag{19}$$

of degree m_k, the set of $\boldsymbol{v}$ resulting unitary zeros of ACP is given as

$$\boldsymbol{\Xi} = \{\boldsymbol{v} \in \boldsymbol{\Lambda} \mid \nu_k(\boldsymbol{v}) \neq m_k/2\} \tag{20}$$

where

$$\boldsymbol{\Lambda} = \left\{\boldsymbol{v} \in \mathbb{R}^{(p-1)} \mid v_{j\neq t} \in [0, 2\pi] \text{ and } v_t \in [0, \pi]\right\}, \quad t \in (1, \ldots, k-1, k+1, \ldots, p) \tag{21}$$

and $\nu_k(\boldsymbol{v}) = \mathrm{Var}\{\sigma^{(\mathrm{k})}_{\mathrm{m_k}-1}(\boldsymbol{v}), \sigma^{(\mathrm{k})}_{\mathrm{m_k}-2}(\boldsymbol{v}), \ldots, \sigma^{(\mathrm{k})}_0(\boldsymbol{v})\}$ such that $\sigma_l^{(k)}(\boldsymbol{v}) = T_l^{(k)}(1, \boldsymbol{v})$ are obtained by the recursion equations in (15)–(17) from $D(z_k, \boldsymbol{v})$.

Proof. Let β denote the number of the unitary zeros ($|z_k| = 1$) of $ACP(z_k, \boldsymbol{v})$ for any $\boldsymbol{v} \in \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is given by (21). Note that we consider only the half of the set $\boldsymbol{\Upsilon}$ in (12), which is represented by the new set $\boldsymbol{\Lambda}$, since the other half of $\boldsymbol{\Upsilon}$ can be achieved by the symmetricity in Lemma 2. From Theorem 3, $ACP(z_k, \boldsymbol{v})$ has at least one unitary zero, i.e. $\beta \neq 0$, if and only if the number of zeros of $\bar{D}(z_k, \boldsymbol{v}) := \partial ACP(z_k, \boldsymbol{v})/\partial z_k$ in $\overline{\mathbb{D}}$ is $\mu_k \neq (m_k - 2)/2$, where m_k denotes the degree of ACP. Then, considering Theorem 2 of the Bistritz method, the number of zeros of $\bar{D}(z_k, \boldsymbol{v})$ in $\overline{\mathbb{D}}$ must be $\mu_k \neq (m_k - 1) - \nu_k$ for any $\boldsymbol{v}$, where ν_k is the number of sign variations in the sequence of $\sigma_l^{(k)}(\boldsymbol{v}) = T_l^{(k)}(1, \boldsymbol{v})$, $l = 0, 1, \ldots, m_k - 1$, which are obtained by the recursion equations in (15)–(17) from $\bar{D}(z_k, \boldsymbol{v})$. Combining two conditions for μ_k, we get $\nu_k(\boldsymbol{v}) \neq m_k/2$ for any $\boldsymbol{v} \in \boldsymbol{\Lambda}$ to result the existence of at least one unitary zero of $ACP(z_k, \boldsymbol{v})$. However, as referred to in Remark 2, the recursion algortihm is to be applied to the polynomial $D(z_k, \boldsymbol{v}) = (z_k - 1)\bar{D}(z_k, \boldsymbol{v})$ since ACP is a polynomial with complex variable coefficients in terms of $\boldsymbol{v}$. The degree of $D(z_k, \boldsymbol{v})$ is m_k and $\sigma^{(k)}_{m_k}(\boldsymbol{v}) \equiv 0$ due to the latter added zero at $z_k = 1$ and the rest of the sequence $\{\sigma^{(k)}_{m_k-1}(\boldsymbol{v}), \sigma^{(k)}_{m_k-2}(\boldsymbol{v}), \ldots, \sigma^{(k)}_0(\boldsymbol{v})\}$ is exactly related to the zeros of $\bar{D}(z_k, \boldsymbol{v}) = \partial ACP(z_k, \boldsymbol{v})/\partial z_k$. □

Notice that the numerical unitary root checking procedure of ACE in step (ii) of the CTCR method given above, is avoided with the Corollary 1 since the algebraic condition for ACP to have a unitary root, i.e. $|z_k^*| = 1$, is derived by means of Bistritz Tabulation. Moreover, the building block parameter ($\boldsymbol{v}$) space, i.e. $[0, 2\pi]^{p-1}$, is reduced by half with Lemma 2 giving the symmetricity property for v_j's, i.e. $[0, 2\pi]^{p-1} \to \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is as in (21). Actually, this symmetry property is a result of the imaginary crossing of CE at $s = i\omega$ for a delay set $\boldsymbol{\tau} \in \wp$ and its conjugate correspondence at $s = -i\omega$ for the same delay set; see Alikoc and Ergenc [2017]. Thus, the computational load for the determination of Ω in (5) is also reduced by half consequently, as in the step (v) of the new CTCR procedure given below.

As a result, the new procedure for CTCR method with EKS approach is presented as follows:

(i) Calculate the $ACP(z_k, \boldsymbol{v})$ in (11) for the system (1) as described in (3) and (10), then derive $D(z_k, \boldsymbol{v})$ given by (19).
(ii) Obtain the set Ξ in (20) which consists only the $\boldsymbol{v}$ value sets resulting unitary zeros, z_k^*, of ACP by applying recursion algorithm (15)–(17) to $D(z_k, \boldsymbol{v})$.
(iii) Solving the roots of ACP for all $\boldsymbol{v} \in \Xi$, achieve the corresponding z_k^* values with unity magnitude.
(iv) Compute the imaginary crossing frequencies (ω) by inserting each $\mathbf{z} \in \mathbb{T}^p$ found in step (iii) and $s = \omega i$ to CE in (2).

(v) Obtain $\boldsymbol{\upsilon} \in \boldsymbol{\Upsilon}$ in $[0, 2\pi]^{p-1}$ applying the symmetry property in Lemma 2, and the corresponding crossing frequencies by inverting the sign of ω found in step (iv). This yields the complete building hypersurface $\boldsymbol{\Delta}$ in (7) and the set of all imaginary crossing frequencies Ω in (5).

(vi) Calculate the switching hypersurfaces of delays described in (6) with the corresponding $\mathbf{z}$ and ω, found in steps (iii)–(v).

(vii) Finally, determine the directions of stability switching on the boundaries of hypersurfaces and NU for the regions in delay space, by root tendency given in (14). The regions with NU=0 are stable.

Note that the determination of the unitary root set of ACE is improved with an algebraic approach, namely Bistritz Tabulation. This actually avoids unnecessary computation of solving the roots of ACP for the building block elements in $[0, 2\pi]^{p-1}$, which do not correspond to a unitary root. By the proposed method, one can find the complete spectral delay space $\boldsymbol{\Upsilon}$ in (12) to accomplish stability posture in delay space in a more computationally efficient way compared to the procedure given in Ergenc et al. [2007]. These advantageous facts are illustrated by applying the new procedure to a previously studied 3-delay system, in the following section. For further discussion on the computational complexity to obtain ACE and ACP, one can refer to Alikoc and Ergenc [2017].

4. CASE STUDY

To compare the proposed DDS method with the former one, we borrow an example from Ergenc et al. [2007]. Consider the system (1) where $n = 2$ and $p = 3$ with the system matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -8 & -3 \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 0 & 0 \\ -1 & -3 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 0 & 0 \\ -8 & 1 \end{bmatrix}, \quad \mathbf{B}_3 = \begin{bmatrix} 0 & 0 \\ -5 & 0 \end{bmatrix}. \tag{22}$$

The characteristic polynomial of the system is

$CE(s, \boldsymbol{\tau}) = s^2 + 3s + 8 + (3s+1)e^{-s\tau_1} + (8-s)e^{-s\tau_2} + 5e^{-s\tau_3}$

which is Hurwitz stable for $\boldsymbol{\tau} = \mathbf{0}$. The corresponding self-inversive auxiliary characteristic polynomial is found by (3) and multiplying it by z_3^2 as in (10), as

$$\begin{aligned} ACP(\mathbf{z}) = {} & 25z_3^4 + z_3^3 \big(140 + 45z_1^{-2} + 125(z_1^{-1} - z_2^{-1}) + 55z_1 \\ & + 5z_2^{-2} - 30z_1^{-1}z_2^{-1} - (15z_1z_2^{-1} + z_2z_1^{-1}) + 65z_2\big) \\ & + z_3^2 \big(178 + 82(z_1^{-2} + z_1^2) + 253(z_1^{-1} + z_1) \\ & + 48(z_2^{-2} + z_2^2) - 23(z_1z_2^{-2} + z_1^{-1}z_2^2) \\ & + 127(z_2^{-1} + z_2) + 37(z_1z_2 + z_1^{-1}z_2^{-1}) \\ & + 143(z_1z_2^{-1} + z_1^{-1}z_2) + 69(z_1^2z_2^{-1} + z_1^{-2}z_2^{-1})\big) \\ & + z_3 \big(140 + 45z_1^2 + 125(z_1 - z_2) + 55z_1^{-1} + 5z_2^2 \\ & - 30z_1z_2 - (15z_1^{-1}z_2 + z_2^{-1}z_1) + 65z_2^{-1}\big) + 25 \end{aligned} \tag{23}$$

of degree $m_3 = 4$, where $z_1 := e^{-i\omega\tau_1}$, $z_2 := e^{-i\omega\tau_2}$, and $z_3 := e^{-i\omega\tau_3}$. One can rewrite the above ACP in the form (11), i.e. $ACP(z_3, \upsilon_1, \upsilon_2)$, by defining $\upsilon_1 := \omega\tau_1$ and $\upsilon_2 := \omega\tau_2$. Applying the recursion algorithm in (15)–(17) to the polynomial $D(z_3, \upsilon_1, \upsilon_2)$ derived as in (19) in Corollary 1, $\sigma_l^{(3)}(\upsilon_1, \upsilon_2) = T_l^{(3)}(1, \upsilon_1, \upsilon_2)$, $l = 0, 1, 2, 3$, are obtained. Then, the subset Ξ in (20) of $\boldsymbol{\Lambda} = \big\{\boldsymbol{\upsilon} \in \mathbb{R}^2 \mid \upsilon_1 \in [0, 2\pi]\ ,\ \upsilon_2 \in [0, \pi]\big\}$ in Corollary 1 ($\nu_3(\boldsymbol{\upsilon}) \neq 2$) is attained. The set $\boldsymbol{\Upsilon} \in [0, 2\pi]^{p-1}$ given by (12) yielding $|z_3| = 1$ is shown in Fig. 1-(a) which is achieved applying the symmetry w.r.t. (π, π) for Ξ, i.e. $\boldsymbol{\Upsilon} = \Xi \cup \Xi_s$ where $\Xi_s = \{\boldsymbol{\upsilon} \in \mathbb{R}^2 \mid \upsilon_1 \in [0, 2\pi]\ ,\ \upsilon_2 \in (\pi, 2\pi],\ \nu_3(\boldsymbol{\upsilon}) \neq 2\}$. Also, the building hypersurface $\boldsymbol{\Delta} = \{\boldsymbol{\upsilon} \in [0, 2\pi]^3 \mid \boldsymbol{\upsilon} = \overline{\arg}(\mathbf{z}), \mathbf{z} \in \mathbf{Z}\}$ where $\mathbf{Z}$ is as in (4), is depicted in Fig. 1-(b). Note that, the number of points (with a sufficient resolution -$\Delta\upsilon_j$-) in (υ_1, υ_2) space for which the zeros of $ACP(z_3, \boldsymbol{\upsilon})$ to be calculated, is reduced to one sixth with the new proposed algebraic approach compared to the former numerical methodology in Ergenc et al. [2007].

The stability posture of the system (1) with the matrices (22) in delay space is reproduced for $\tau_3 = 2$ s and for $\tau_3 = 2.5$ s in Fig. 2. The cross-sections of kernel and offspring hypercurves with constant τ_3 values are shown in red and black, respectively. The number of unstable roots, which are determined by the invariance property of the root tendency (14) on switching hypersurfaces, are indicated for some of the regions generated by these hypercurves. The stable regions where NU=0, are shaded. The stability outlook matches with that of Ergenc et al. [2007] precisely.

5. CONCLUSION

A well-known CTCR procedure via EKS to obtain stability maps in multiple-delay space is improved by checking the sign variations instead of solving polynomials to find unitary roots of ACP, and by using the symmetricity property of the spectral delay space, which reduces the computational load. The numerical procedure in the previous methodology has been significantly reduced to determine the switching hypersurfaces in delay space. The possible direction of the subsequent research will be the extraction of the stability regions in the space of system parameters and delays together.

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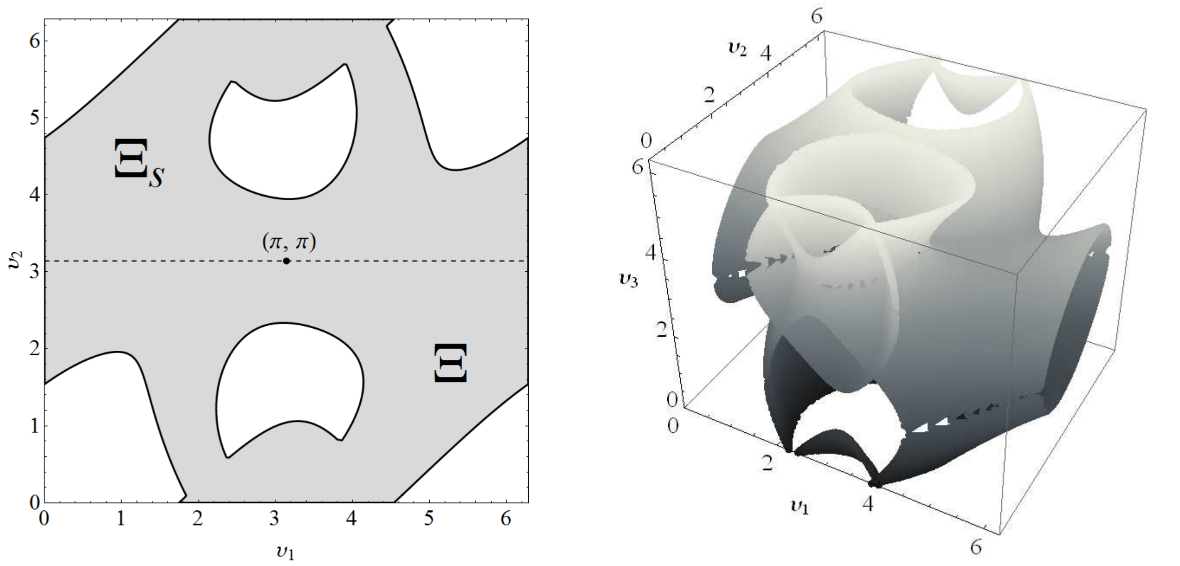
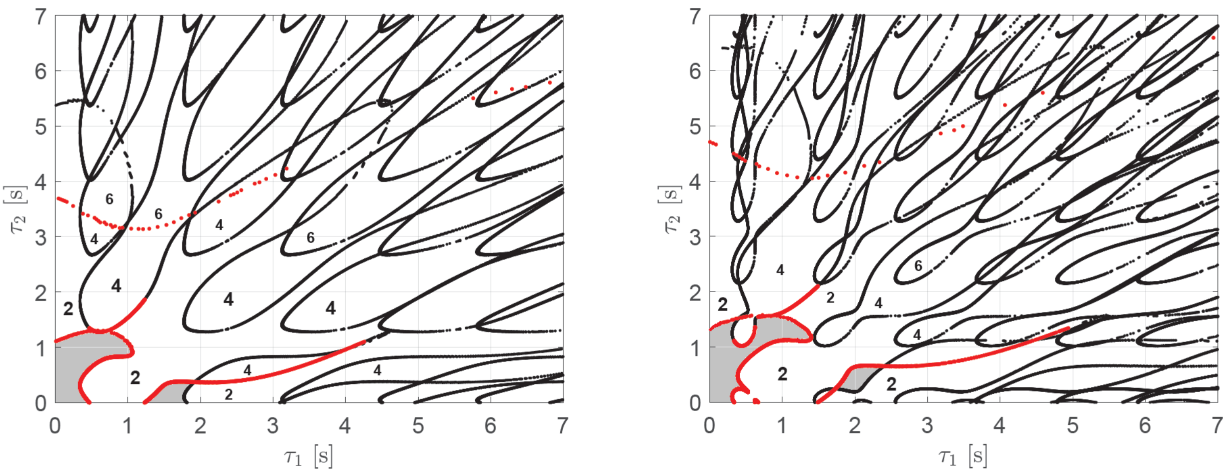


(a) **(b)**

Fig. 1. **(a)** Ξ in (20) obtained by the application of Corollary 1, Ξ_s obtained by the application of Lemma 2, and entire set $\boldsymbol{\Upsilon} = \Xi \cup \Xi_s$ in (12) for the system (1) with (22). **(b)** The building hypersurface $\boldsymbol{\Delta}$ in (7) for (1) with (22)



(a) **(b)**

Fig. 2. Stability map of the system (1) with (22) in (τ_1, τ_2) space for $\tau_3 = 2$ s **(a)** and $\tau_3 = 2.5$ s **(b)**. The cross-sections of kernel and offspring hypersurfaces are in red and black, respectively. Stable regions (NU=0) are shaded.

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