At the junction of Lyapunov-Krasovskii and Razumikhin approaches

Irina V. Alexandrova, * **Alexey P. Zhabko** *

* *Saint Petersburg State University, Saint Petersburg, Russia (e-mail: i.medvedeva@spbu.ru, zhabko.apmath.spbu@mail.ru)*

Abstract: In this paper, a modification of the Krasovskii theorem for a nonlinear class of systems is presented. The idea is to replace the positive definiteness condition of the functional with this condition on the special Razumikhin-type set of functions only while retaining the other classical conditions. The result is motivated by the fact that this idea leads to the necessary and sufficient stability condition for linear time-invariant systems. Moreover, this condition is constructive and allows us not only to directly analyze the stability but also to find the robustness bounds on the matrix parameters and on the delays and to construct the exponential estimates for solutions. An overview of these results for linear systems is also presented.

Keywords: time-delay systems, asymptotic stability, exponential stability, robust stability, Lyapunov–Krasovskii functionals, Razumikhin condition, uncertain delay

1. INTRODUCTION

The Lyapunov–Krasovkii functionals approach is the most powerful tool in the stability analysis of time-delay systems, especially in a nonlinear case or when time-varying perturbations in the linear systems are studied. In this approach, the criteria for stability are expressed in terms of some properties of the functionals and their derivatives along the solutions of a system. In particular, a linear system is exponentially stable, if and only if there exists a positive definite functional with a negative definite derivative along the solutions, as stated in the well-known Krasovskii theorem.

For linear systems, Krasovskii theorem has motivated an idea to use the functionals with prescribed derivative which was developed in Repin (1965) and Infante & Castelan (1978). In later works, the derivative was set either as a negative definite quadratic form of the vector $x(t)$ (Huang, 1989) or as a negative definite quadratic functional of the true state x_t (Kharitonov & Zhabko, 2003). Although in both cases the obtained functional is positive definite if the system is exponentially stable, only the second one, which was called the functional of complete type, admits a quadratic lower bound and, therefore, allows us to solve the various problems in applications. In particular, complete-type functionals make it possible to obtain the exponential estimates for solutions (Kharitonov & Hinrichsen, 2004), to analyze the robustness with respect to the coefficients (Kharitonov & Zhabko, 2003) and the delays (Kharitonov & Niculescu, 2003), to find the critical values of delay (Ochoa et. al, 2013), to compute the norm of the transfer matrix (Jarlebring et al., 2011), etc. In Egorov & Mondie (2013) and Egorov et al. (2017), an interesting approach allowing to apply these functionals directly to the stability analysis is presented.

In this paper, we present a modification of the Krasovskii theorem for a general class of systems. The main idea is to require a functional to be positive definite on the set of functions satisfying the Razumikhin-type condition (see Razumikhin, 1956) only instead of the set of all functions, which, in a combination with the negativeness of the derivative and the other classical conditions, leads to a sufficient condition of asymptotic stability. The linear case serves as a motivation for such formulation, since the condition becomes necessary and sufficient for linear systems (Medvedeva & Zhabko, 2015b) and gives a constructive technique for the stability analysis (Medvedeva & Zhabko, 2013). Moreover, the functional with a derivative prescribed as $-x^T(t)Wx(t)$, where W is a positive definite matrix (see Huang, 1989), is shown to admit a quadratic lower bound on the abovementioned Razumikhin-type set of functions (Medvedeva & Zhabko, 2015b), and, therefore, to be effective in applications along with the complete-type functional.

The paper is organized as follows. Sections 2 and 3 are dedicated to a nonlinear class of systems and state the main result. Subsequent sections give an overview on the joint Lyapunov–Krasovskii and Razumikhin approach for linear time-invariant systems. In particular, in Section 4 we present the exponential stability and instability criteria. In Section 5, we describe the methodology for the stability analysis which is based on the above criteria. An important point is that we show how the obtained sufficient stability condition is connected with the necessary one. In Section 6, we use our methodology to obtain the stability regions in the space of parameters in a number of examples. Finally, in Section 7 we show that our approach makes it possible to apply the functional from Huang (1989) to the robust stability analysis with respect to unknown parameters in the matrices, to analysis of systems with uncertain delays and to construction of the exponential estimates for solutions.

Notation. $PC\big([-h,0],\mathbb{R}^n\big)$ stands for the space of the piecewise continuous functions defined on $[-h,0]$ with the norm $\|\varphi\|_h = \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|$, where $\|\cdot\|$ is the Euclidian norm; $x(t,t_0,\varphi)$ denotes the solution of a time-delay system with an initial time instant $t = t_0$ and an initial function φ; x_t stands for the state of a time-delay system, i.e. the function $x(t+\theta)$, $\theta \in [-h,0]$; 0_h is the zero function: $0_h(\theta) = 0$, $\theta \in [-h,0]$.

2. PRELIMINARIES

Consider a time-delay system

$$\dot{x}(t) = f(t, x_t), \tag{1}$$

where the functional $f(t,\varphi)$ is defined for $t \geqslant 0$ and $\varphi \in PC\big([-h,0],\mathbb{R}^n\big)$ and satisfies the following conditions:

A. It is continuous in both arguments;

B. $\forall H > 0\ \exists M(H) > 0$ such that $\|f(t,\varphi)\| \leqslant M(H)$ $\forall t \geqslant 0$, $\forall \varphi \in PC\big([-h,0],\mathbb{R}^n\big)$ with $\|\varphi\|_h \leqslant H$;

C. $\forall H > 0\ \exists L(H) > 0$ such that

$$\|f(t,\varphi) - f(t,\psi)\| \leqslant L(H)\|\varphi - \psi\|_h$$

$\forall t \geqslant 0$, $\forall \varphi,\psi \in PC\big([-h,0],\mathbb{R}^n\big)$ such that $\|\varphi\|_h \leqslant H$, $\|\psi\|_h \leqslant H$;

D. $f(t,0_h) \equiv 0\ \forall t \geqslant 0$, i.e. system (1) has a trivial solution.

Assumptions A–C ensure the uniqueness and continuity of a solution for every initial instant $t_0 \geqslant 0$ and every initial function $\varphi \in PC\big([-h,0],\mathbb{R}^n\big)$, see Kharitonov (2013), p. 6.

The basic stability definitions are recalled below, see, for instance, Kharitonov (2013). The trivial solution of system (1) is said to be stable (in the Lyapunov sense), if $\forall t_0 \geqslant 0$, $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall \varphi \in PC\big([-h,0],\mathbb{R}^n\big)$ with $\|\varphi\|_h < \delta$ we have $\|x(t,t_0,\varphi)\| < \varepsilon\ \forall t \geqslant t_0$. The trivial solution of system (1) is asymptotically stable, if, in addition, $\exists \Delta > 0$ such that $\|x(t,t_0,\varphi)\| \xrightarrow[t\to+\infty]{} 0$, if $\|\varphi\|_h < \Delta$. The trivial solution of (1) is exponentially stable, if there exist $\Delta_0 > 0$, $\gamma \geqslant 1$ and $\sigma > 0$ such that

$$\|x(t,t_0,\varphi)\| \leqslant \gamma e^{-\sigma(t-t_0)}\|\varphi\|_h, \quad t \geqslant t_0, \tag{2}$$

$\forall t_0 \geqslant 0$, $\forall \varphi \in PC\big([-h,0],\mathbb{R}^n\big)$ with $\|\varphi\|_h < \Delta_0$. A function $v_1(x)$ ($\|x\| \leqslant H$) is called positive definite, if it is continuous, and $v_1(x) > 0$ for $x \neq 0$, $v_1(0) = 0$.

3. MAIN RESULT

Introduce the following Razumikhin-type set of functions:

$$S = \big\{\varphi \in PC\big([-h,0],\mathbb{R}^n\big) \big| \, \|\varphi\|_h = \|\varphi(0)\| \leqslant H\big\}.$$

Theorem 1. The trivial solution of system (1) is asymptotically stable, if there exists a functional $v(t,\varphi)$ such that the following conditions hold:

1. The functional is continuous in φ at point $\varphi = 0_h$ uniformly with respect to $t \geqslant 0$, and $v(t,0_h) = 0\ \forall t \geqslant 0$;

2. There exist $H > 0$ and a positive definite function $v_1(x)$, $\|x\| \leqslant H$, such that

$$v(t,\varphi) \geqslant v_1(\varphi(0)),\ t \geqslant 0,\ \varphi \in S.$$

3. The functional is differentiable along the solutions of system (1), and

$$\frac{\mathrm{d}v(t,x_t)}{\mathrm{d}t} \leqslant -w(x(t)), \quad t \geqslant 0,$$

where $w(x)$, $\|x\| \leqslant H$, is a positive definite function.

Proof. The proof is based on the proofs of the classical results on the Lyapunov stability, see Theorems 1.4 and 1.8 in Kharitonov (2013).

Part 1. In this part, we prove that the trivial solution of system (1) is stable. Suppose, by contradiction, that there exist $t_0 \geqslant 0$ and $\varepsilon_1 > 0$ ($\varepsilon_1 < H$) such that for any $\delta > 0$ there exist an initial function $\varphi \in PC\big([-h,0],\mathbb{R}^n\big)$, $\|\varphi\|_h < \delta$, and $t_1 \geqslant t_0$ for which $\|x(t_1,t_0,\varphi)\| \geqslant \varepsilon_1$. Set

$$\lambda = \min_{\|x\|=\varepsilon_1} v_1(x) > 0.$$

By continuity of functional $v(t_0,\varphi)$ at point $\varphi = 0_h$, there exists $\delta_1 > 0$ such that $v(t_0,\varphi) < \lambda$, if $\|\varphi\|_h < \delta_1$. Assume that $\delta_1 > \varepsilon_1$. Then, take an initial function $\varphi \in S$ such that $\|\varphi(0)\| = \|\varphi\|_h = \varepsilon_1$. For this function, $v(t_0,\varphi) \geqslant v_1(\varphi(0)) \geqslant \lambda$ by the second condition of the theorem. From the other hand, since $\|\varphi\|_h < \delta_1$, we have $v(t_0,\varphi) < \lambda$. The obtained contradiction proves that $\delta_1 \leqslant \varepsilon_1$.

According to our assumption, there exist an initial function $\widetilde{\varphi} \in PC\big([-h,0],\mathbb{R}^n\big)$, $\|\widetilde{\varphi}\|_h < \delta_1$, and $t_1 \geqslant t_0$ such that $\|x(t_1,t_0,\widetilde{\varphi})\| \geqslant \varepsilon_1$. Due to continuity of the solution, there exists a time instant $t^* \geqslant t_0$ such that $\|x(t^*,t_0,\widetilde{\varphi})\| = \varepsilon_1$ and $\|x(t,t_0,\widetilde{\varphi})\| < \varepsilon_1$ for $t < t^*$. Hence, $x_{t^*} \in S$, and

$$v_1(x(t^*,t_0,\widetilde{\varphi})) \leqslant v(t^*, x_{t^*}(t_0,\widetilde{\varphi})) \leqslant v(t_0,\widetilde{\varphi}),$$

the last due to the third condition of the theorem. From the other hand,

$$v_1(x(t^*,t_0,\widetilde{\varphi})) \geqslant \lambda,$$

hence $v(t_0,\widetilde{\varphi}) \geqslant \lambda$. The contradiction ends the proof of the first part.

Part 2. In this part, we prove that the trivial solution of system (1) is asymptotically stable. By the first condition of the theorem, there exists $\eta > 0$ such that for every initial function $\varphi \in PC\big([-h,0],\mathbb{R}^n\big)$, $\|\varphi\|_h < \eta$, we get $|v(t,\varphi)| < B$ for any $t \geqslant 0$, where $B > 0$. As previously proved, for a $t_0 \geqslant 0$ there exists $\delta > 0$ such that for every initial function $\varphi \in PC\big([-h,0],\mathbb{R}^n\big)$, $\|\varphi\|_h < \delta$, we have $\|x(t,t_0,\varphi)\| < \eta$ for any $t \geqslant t_0$. Assume that $\|\varphi\|_h < \delta$. We need to prove that $\|x(t,t_0,\varphi)\| \xrightarrow[t\to+\infty]{} 0$. Suppose, by contradiction, that there exists a $\beta > 0$ and a sequence $\{t_j\}_{j=1}^{+\infty}$ such that $|t_{j+1} - t_j| \geqslant h$ for every j and $\|x(t_j,t_0,\varphi)\| \geqslant \beta$, $j = 1,2,\ldots$

First we prove that there exists a $\tau > 0$ such that

$$\|x(t,t_0,\varphi)\| \geqslant \frac{\beta}{2}, \quad t \in [t_j, t_j+\tau], \quad j = 1,2,\ldots \tag{3}$$

Indeed, let $t \in [t_j, t_j+\tau]$. Then,

$$x(t,t_0,\varphi) = x(t_j,t_0,\varphi) + \int_{t_j}^{t} f(s, x_s(t_0,\varphi))ds,$$

and hence

$$\|x(t,t_0,\varphi) - x(t_j,t_0,\varphi)\| \leqslant M(\eta)(t-t_j) \leqslant M(\eta)\tau.$$

This implies that

$$\|x(t,t_0,\varphi)\| \geqslant \|x(t_j,t_0,\varphi)\| - M(\eta)\tau \geqslant \frac{\beta}{2},$$

if $\tau \leqslant \dfrac{\beta}{2M(\eta)}$. We set $\tau = \min\{\dfrac{\beta}{2M(\eta)}, h\}$, then (3) holds and different intervals $[t_j, t_j+\tau]$ do not have common points.

Integrating the third condition of the theorem, we obtain

$$v(t_0,\varphi) \geqslant v(t, x_t(t_0,\varphi)) + \int_{t_0}^{t} w(x(s,t_0,\varphi))ds.$$

Here $v(t, x_t(t_0,\varphi)) > -B$, as $\|x_t(t_0,\varphi)\|_h < \eta$ for any $t \geqslant t_0$. Next,

$$\int_{t_0}^{t} w(x(s,t_0,\varphi))ds \geqslant \sum_{j=1}^{N(t)} \int_{t_j}^{t_j+\tau} w(x(s,t_0,\varphi))ds \geqslant \tau\gamma N(t),$$

where

$$\gamma = \min_{\frac{\beta}{2}\leqslant\|x\|\leqslant\eta} w(x) > 0,$$

and $N(t)$ is a number of intervals $[t_j, t_j+\tau] \subset [0,t]$ $(j = 1,2,\ldots)$, $N(t) \xrightarrow[t\to+\infty]{} +\infty$. Hence,

$$v(t_0,\varphi) > -B + \tau\gamma N(t) \xrightarrow[t\to+\infty]{} +\infty,$$

and we arrive at the contradiction. □

4. STABILITY THEOREMS FOR LINEAR SYSTEMS

4.1 Exponential Stability Criterion

There is a certain problem to converse Theorem 1 since there is no functional corresponding to the necessary and sufficient stability conditions in the general case. However, the result admits a conversion for a linear system of the form

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t-h_j), \tag{4}$$

where A_j are the constant matrices, and $0 = h_0 < h_1 < \ldots < h_m = h$ are the constant delays, for which the following criterion is satisfied. We consider system (4) in the remaining of the paper.

Theorem 2. (Medvedeva & Zhabko, 2015b) Given a positive definite matrix W, system (4) is exponentially stable, if and only if there exists a functional $v_0(\varphi)$ such that the following conditions hold:

1. $\dfrac{dv_0(x_t)}{dt} = -x^T(t)Wx(t)$ along the solutions of (4);
2. there exists a $\mu > 0$ such that
$$v_0(\varphi) \geqslant \mu\|\varphi(0)\|^2$$

for every function $\varphi \in S$, i.e. such that $\|\varphi(0)\| = \|\varphi\|_h$.

Remark 3. As it follows from the proof, Theorem 2 remains true if we add the following conditions on the derivative to the set S :

$$\|\varphi^{(k)}(\theta)\| \leqslant \left(\sum_{j=0}^{m}\|A_j\|\right)^k \|\varphi(0)\|, \theta \in [-h,0],\ k = 1,2,\ldots$$

It is well-known (see Kharitonov & Zhabko, 2003) that the functional satisfying the first condition of Theorem 2 is of the form

$$\begin{aligned} v_0(x_t) &= x^T(t)U(0)x(t) \\ &\quad + 2x^T(t)\sum_{j=1}^{m}\int_{-h_j}^{0} U(-\theta-h_j)A_j x(t+\theta)d\theta \\ &\quad + \sum_{k=1}^{m}\sum_{j=1}^{m}\int_{-h_k}^{0} x^T(t+\theta_1)A_k^T \\ &\times\left(\int_{-h_j}^{0} U(\theta_1+h_k-\theta_2-h_j)A_j x(t+\theta_2)d\theta_2\right)d\theta_1, \end{aligned} \tag{5}$$

where $U(\tau)$, $\tau \in [-h,h]$, is the Lyapunov matrix associated with W, i.e. the matrix satisfying the following equations:

$$\begin{gathered} U'(\tau) = \sum_{j=0}^{m} U(\tau-h_j)A_j,\ \tau \geqslant 0; \\ U(-\tau) = U^T(\tau),\ \tau \geqslant 0; \\ \sum_{j=0}^{m}[U(-h_j)A_j + A_j^T U^T(-h_j)] = -W. \end{gathered} \tag{6}$$

The Lyapunov matrix exists and is unique, if system (4) satisfies the Lyapunov condition, i.e. the system has no eigenvalue s such that $-s$ is also an eigenvalue, see Kharitonov (2013).

4.2 Instability Criterion

For the sake of completeness, we present the criterion for instability here.

Theorem 4. (Zhabko & Medvedeva, 2016) Let system (4) satisfy the Lyapunov condition. Given a positive definite matrix W, system (4) is unstable, if and only if there exists a functional $v_0(\varphi)$ such that the following conditions hold:

1. $\dfrac{dv_0(x_t)}{dt} = -x^T(t)Wx(t)$ along the solutions of (4);
2. there exist a $\mu > 0$ and a function $\widetilde{\varphi} \in S$ such that
$$v_0(\widetilde{\varphi}) \leqslant -\mu\|\widetilde{\varphi}(0)\|^2.$$

5. METHODOLOGY FOR STABILITY ANALYSIS

Theorem 2 opens a constructive way to analyze the stability of system (4). To avoid cumbersome computations, we describe methodology for the case $m = 1$. First of all, divide the interval $[-h,0]$ into N equal parts by the points $\theta_j = -j\Delta$, $j = \overline{0,N}$, where $\Delta = h/N$. Then, take a function $\varphi \in S$ and construct for it the spline approximation $q(\theta)$, $\theta \in [-h,0]$, satisfying the equalities:

$$q(\theta_j) = \varphi(\theta_j), \quad q'(\theta_j) = \varphi'(\theta_j), \quad j = \overline{0,N}.$$

These equalities ensure the smoothness of the spline, and we choose the spline consisting of cubic polynomials, as there are 4 conditions at each interval of partition. Direct calculations lead to the following formula:

$$\begin{aligned} q(s+\theta_j) &= g_1(s)\varphi(\theta_j) + g_2(s)\varphi(\theta_{j+1}) + g_3(s)\varphi'(\theta_j) \\ &\quad + g_4(s)\varphi'(\theta_{j+1}),\ s \in [-\Delta,0],\ j = \overline{0,N-1}, \end{aligned}$$

where

$$g_1(s) = -(2s+3\Delta)s^2/\Delta^3 + 1,\ g_2(s) = 1 - g_1(s),$$
$$g_3(s) = (s+\Delta)f(s),\ g_4(s) = sf(s),\ f(s) = (s+\Delta)s/\Delta^2.$$

By Taylor formula, using the constraint on 4th derivative of φ, see Remark 3, we estimate an approximation error:

$$\begin{gathered} \big\|\eta(s+\theta_j)\big\| = \big\|\varphi(s+\theta_j) - q(s+\theta_j)\big\| \leqslant C\,\chi(s)\|\varphi(0)\|, \\ C = \frac{K\sqrt{n}}{24},\ K = \|A_0\| + \|A_1\|, \\ \chi(s) = s^4 + 6s^3\Delta + 7s^2\Delta^2,\ s \in [-\Delta,0],\ j = \overline{0,N-1}. \end{gathered} \tag{7}$$

Next, substitute approximation q into the functional and take the error into account. For example, for the second summand of the functional we have

$$2\varphi^T(0)\int_{-h}^{0} U^T(\theta+h)A_1\varphi(\theta)d\theta$$

$$= 2\varphi^T(0)\sum_{j=1}^{N}\int_{-\Delta}^{0} U^T(s+j\Delta)A_1\varphi(s+\theta_{N-j})ds$$

$$= 2\varphi^T(0)\sum_{j=1}^{N}\Bigg[L^{1j}\varphi(\theta_{N-j}) + L^{2j}\varphi(\theta_{N-j+1})$$

$$+L^{3j}\varphi'(\theta_{N-j}) + L^{4j}\varphi'(\theta_{N-j+1})\Bigg] + \Upsilon_1, \text{ where}$$

$$L^{kj} = \int_{-\Delta}^{0} g_k(s)U^T(s+j\Delta)dsA_1, \;\; k=\overline{1,4},$$

$$\Upsilon_1 = 2\varphi^T(0)\sum_{j=1}^{N}\int_{-\Delta}^{0} U^T(s+j\Delta)A_1\eta(s+\theta_{N-j})ds.$$

Applying formula (7), we obtain

$$|\Upsilon_1| \leqslant 2C\|A_1\|\sum_{j=1}^{N}\int_{-\Delta}^{0}\|U(s+j\Delta)\|\chi(s)ds\|\varphi(0)\|^2.$$

Transforming the other summands of functional (5) in the same way, we get the following:

$$v_0(\varphi) = \Lambda(p,\widehat{\varphi}) + \Upsilon,$$

where $\Lambda(p,\widehat{\varphi})$ is a quadratic form, $p=\varphi(0)$, and $\widehat{\varphi}$ is the vector obtained by a concatenation of vectors $\varphi(\theta_j)$ for $j=\overline{1,N}$ and $\varphi'(\theta_j)$ for $j=\overline{0,N}$, the total dimension of $\widehat{\varphi}$ is $(2N+1)n$. The variable Υ denotes the group of summands depending on the error of approximation $\eta(\theta)$ for which there exists a bound of the form

$$\Upsilon \geqslant -\delta\|p\|^2, \text{ where } \delta > 0.$$

Hence, functional (5) admits a quadratic lower bound

$$v_0(\varphi) \geqslant \Lambda(p,\widehat{\varphi}) - \delta\|p\|^2, \quad \varphi\in S.$$

According to Theorem 2 and Remark 3 for $k=1$, we get the following stability condition: If there exists $N\in\mathbb{N}$ such that

$$\min_{\substack{\|p\|=1\\ |\widehat{\varphi}_j|\leqslant 1,\, j=\overline{1,nN},\\ |\widehat{\varphi}_j|\leqslant K,\, j=\overline{nN+1,(2N+1)n}}} \Lambda(p,\widehat{\varphi}) - \delta > 0, \tag{8}$$

then system (4) ($m=1$) is exponentially stable, here $\widehat{\varphi}_j$ denotes jth component of the vector $\widehat{\varphi}$.

Remark 5. It is worthy of mention that the value of δ tends to zero as $N\to+\infty$, and besides that, the minimum of Λ in (8) is always positive, if the system is exponentially stable. Hence, the value of N such that (8) is positive knowingly exists for an exponentially stable system.

Remark 6. It follows from the above that

$$v_0(\varphi) \leqslant \Lambda(p,\widehat{\varphi}) + \delta\|p\|^2.$$

Hence, if we replace "$-$" with "$+$" and ">0" with "<0" in (8), then we obtain a sufficient condition for instability in accordance with Theorem 4.

Remark 7. Coefficients of the quadratic form $\Lambda(p,\widehat{\varphi})$ and the parameter δ depend on the Lyapunov matrix defined by equations (6). However, there is no effective technique to compute this matrix in the case when the delays h_j are noncommensurate in the present time, except for a numerical approach for exponentially stable systems in Egorov & Kharitonov (2016). In Zhabko & Medvedeva (2016), a modification of the methodology of Section 5 for this case which is based on a modification of functional (5) is presented.

6. EXAMPLES

In each example, we set N, and check stability condition (8) at the points of a grid in a region at the space of parameters of the system. The points where (8) holds along with the curves of D-subdivision are depicted on figures. To compute the Lyapunov matrix, the semianalytic method (see Kharitonov, 2013) with $W=I$ is used.

Example 8. In Egorov & Mondie (2013), the stability region for equation

$$\dot{x}(t) = -2x(t) + ax(t-1) + bx(t-2)$$

was analyzed. The region obtained by the methodology of Section 5 with $N=25$ is presented on Fig. 1. Note that there are some points which are located within the stability region but are not identified by our approach, as for these points a larger value of N is required. For example, for the point $a=4$, $b=-7/3$ minimum (8) becomes positive with $N=34$.

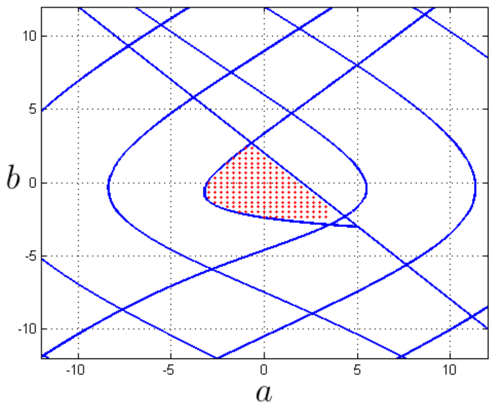

Fig. 1. Example 8, stability region obtained with $N=25$

Example 9. Consider an equation

$$\dot{x}(t) = -x(t) + x(t-h_1) - x(t-h_2),$$

for which the stability regions at the space of delays h_1 and h_2 were studied in Cicco et al. (2011). The results of verification of condition (8) together with the results of Cicco et al. (2011) are represented on Fig. 2.

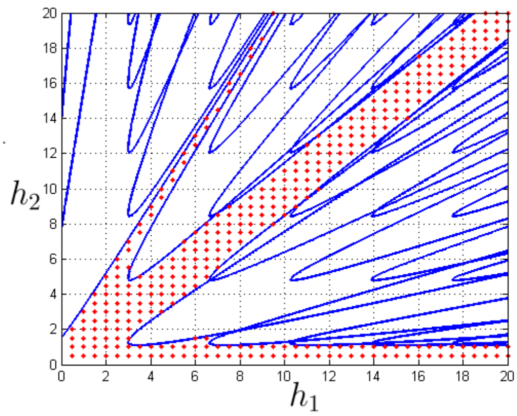

Fig. 2. Example 9, stability region obtained with $N=20$

Example 10. The classical well-known linearized equation of motion of the inverted pendulum is of the form

$$\ddot{\varphi} - \frac{g}{l}\varphi = \frac{1}{l}\ddot{u},$$

where φ is the angle of the pendulum deviation from the vertical line, l is the length of the pendulum, and g is the gravitational acceleration. Consider the simplest control law

$$\ddot{u}(t) = c_1\varphi(t-h) + c_2\dot{\varphi}(t-h) - \varepsilon_1 u(t) - \varepsilon_2 \dot{u}(t),$$

which represents acceleration of pivot of the pendulum, here $\varepsilon_1, \varepsilon_2 > 0$ are chosen sufficiently small so that the exponential stability of the closed-loop system of the 4th order is equivalent to the exponential stability of

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ g/l & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ c_1/l & c_2/l \end{pmatrix} x(t-h), \qquad (9)$$

where $x = \left(\varphi, \dot{\varphi}\right)^T$. We choose $c_1 = -10$, $c_2 = -12$, and analyze the stability of system (9) in the space of parameters l and h. It seems that the stability region obtained by condition (8) with $N = 20$ coincides with the exact one, see Fig. 3. In Table 1, we set $l = 3$, and

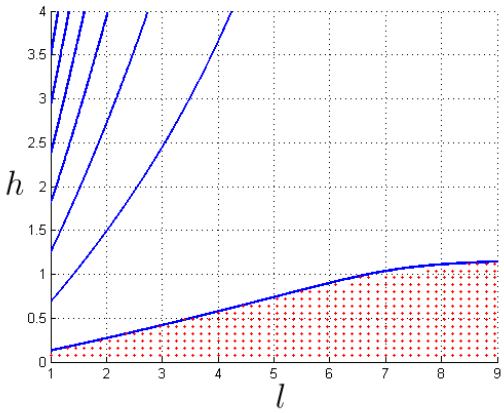

Fig. 3. Example 10, stability region obtained with $N = 20$

find the values h_N such that minimum (8) is positive for $h \in [0, h_N]$ with different values of N. We conclude that the sequence of values h_N converge to the exact critical delay value with increase of N.

Table 1. Example 10, stability boundaries with respect to delay for $l = 3$

N	1	5	10	15	20	24
h_N	0.079	0.175	0.289	0.390	0.420	0.423

7. APPLICATIONS

In this section, we present a number of applications of Theorem 2. Note that we use functional (5), and there is no need in the functionals of complete type in our approach.

7.1 Robustness Bounds

Together with system (4), consider a perturbed system

$$\dot{y}(t) = \sum_{j=0}^{m} (A_j + \Delta_j) y(t-h_j), \qquad (10)$$

here Δ_j are the constant perturbation matrices.

Theorem 11. (Medvedeva & Zhabko, 2015a) Let system (4) be exponentially stable. If

$$\sum_{j=0}^{m} \|\Delta_j\| < \frac{\lambda_{\min}(W)}{2\alpha M}, \qquad (11)$$

where $M = \|U(0)\|$, $\alpha = 1 + \sum_{j=1}^{m} \|A_j\| h_j$, then system (10) remains exponentially stable.

Remark 12. Note that $M = \max_{\tau\in[0,h]} \|U(\tau)\| = \|U(0)\|$, see Egorov & Mondie (2015).

An idea of the proof of Theorem 11 is as follows. First, as in the classical approach, we differentiate original functional (5) along the solutions of system (10):

$$\frac{dv_0(y_t)}{dt} = -y^T(t)Wy(t) + l(y_t), \qquad (12)$$

where

$$l(y_t) = 2\left[\sum_{j=0}^{m} \Delta_j y(t-h_j)\right]^T \omega(y_t), \quad \text{and}$$

$$\omega(y_t) = U(0)y(t) + \sum_{k=1}^{m} \int_{-h_k}^{0} U(-\theta-h_k)A_k y(t+\theta)d\theta,$$

see Kharitonov & Zhabko (2003). Then, we obtain the following estimate:

$$\int_0^t \left(y^T(s)Wy(s) - l(y_s)\right)ds \geqslant \rho \int_0^t \|y(s)\|^2 ds - \Psi, \quad (13)$$

where Ψ is a bounded expression which depends on the initial function and does not depend on t. It is proven that $\rho > 0$, which is equivalent to (11), is a sufficient condition for the exponential stability of system (10). In other words, we replace the classical condition, namely, the negativeness of the derivative of functional along the solutions of perturbed system, with the negativeness of the "principal" part of the integral of this derivative.

7.2 Analysis of Systems With Uncertain Delays

Here we consider a perturbed system of the form

$$\dot{y}(t) = \sum_{j=0}^{m} A_j y(t-h_j-\eta_j), \quad t \geqslant 0, \qquad (14)$$

where η_j are the constant delay perturbations.

Theorem 13. (Alexandrova & Zhabko, 2018) Let system (4) be exponentially stable. If $\eta_j \geqslant -h_j$, $j = \overline{0,m}$, and

$$\sum_{j=0}^{m} |\eta_j|\|A_j\| < \frac{\lambda_{\min}(W)}{2\alpha MK},$$

where $K = \sum_{j=0}^{m} \|A_j\|$, α and M are defined in Theorem 11, then system (14) remains exponentially stable.

This result is also based on formulae (12) and (13), with the difference in the functional

$$l(y_t) = 2\sum_{j=0}^{m} \Big(y(t-h_j-\eta_j) - y(t-h_j)\Big)^T A_j^T \omega(y_t)$$

and some technical details. It is worthy of mention that Theorems 11 and 13 both can be extended to the case of time-varying perturbations.

An important point is that an iterative application of Theorem 13 with respect to one parameter leads to the critical value of this parameter. For example, set $h_j = \gamma_j \mathfrak{h}$, where $\mathfrak{h} \geqslant 0$ is a basic delay, and consider the perturbations of the form $\eta_j = \gamma_j \beta$, $j = \overline{0,m}$, denote $\alpha = \alpha(\mathfrak{h})$ and $M = M(\mathfrak{h})$. Let system (4) be exponentially stable for $\mathfrak{h} = \mathfrak{h}_0$. Introduce the quantities

$$\beta_k = \lambda_{\min}(W)\left[2\alpha(\mathfrak{h}_{k-1})M(\mathfrak{h}_{k-1})K\sum_{j=0}^{m}\gamma_j\|A_j\|\right]^{-1} - \varepsilon_k,$$

$$\mathfrak{h}_k = \mathfrak{h}_0 + \sum_{j=1}^{k}\beta_j, \quad k = 1, 2, \dots$$

with $\varepsilon_k > 0$ for every k, $\varepsilon_k \xrightarrow[k\to+\infty]{} 0$. Then, as shown in Alexandrova & Zhabko (2018),

$$\lim_{k\to+\infty}\mathfrak{h}_k = \bar{\mathfrak{h}},$$

where $\bar{\mathfrak{h}}$ is the critical value of basic delay, i.e. the value under which the exponential stability is lost by the system.

7.3 Exponential Estimates

This section is devoted to estimation of the decay rate σ and the γ-factor from formula (2) ($t_0 = 0$) for an exponentially stable system (4). To estimate the decay rate, we make the change of variable $y(t) = e^{\sigma t}x(t)$, where $\sigma > 0$, then

$$\dot{y}(t) = (A_0 + \sigma I)y(t) + \sum_{j=1}^{m} e^{\sigma h_j}A_j y(t - h_j). \tag{15}$$

System (15) can be perceived as a perturbed system of the form (10), and, as a direct consequence of Theorem 11, we obtain that system (15) is exponentially stable, if

$$R_\sigma = \sigma + \sum_{j=1}^{m}(e^{\sigma h_j} - 1)\|A_j\| < \frac{\lambda_{\min}(W)}{2\alpha M}. \tag{16}$$

The σ such that (16) holds thus constitutes a lower bound for the decay rate of system (4).

Theorem 14. (Medvedeva & Zhabko, 2015a) Let system (4) be exponentially stable. Then, for every solution of (4),

$$\|x(t,\varphi)\| \leqslant \sqrt{\frac{\nu_\sigma}{\mu}}e^{-\sigma t}\|\varphi\|_h, \quad t \geqslant 0.$$

Here σ is such that inequality (16) holds,

$$\nu_\sigma = M\Big(\alpha^2 + \alpha\sum_{j=1}^{m}(e^{\sigma h_j} - 1)\|A_j\|h_j + R_\sigma\sum_{j=1}^{m}\|A_j\|h_j^2\Big),$$

$\mu = \lambda_{\min}(W)\delta/4$, where $\delta > 0$ is a solution of equation

$$\alpha K e^{K\delta} = 1/2\delta.$$

Note that, as for the case of perturbations in delays, we are able to construct the sequence of estimations $\{\sigma_k\}$ converging with $k \to +\infty$ to the exact value of decay rate $\bar{\sigma}$ applying (16) iteratively (see Medvedeva & Zhabko, 2015a, for the details), here $\bar{\sigma}$ is the spectral abscissa of the system. However, increase of σ will lead to increase of ν_σ, i.e. to increase of the γ-factor.

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