Lyapunov functionals and matrices: An overwiew

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Lyapunov-Krasovskii functionals that originally were introduced for the stability analysis of solutions of time-delay systems are recognized now as a powerful tool both for analysis and design of the systems [25], [24]. The variety of problems where the functionals are effectively used includes robustness analysis, stabilization and design of optimal controls to mention just a few of them [35], [28], [20], [10], [15], [26], [27]. Similar to the case of delay free systems the success in application of the functionals depends on the availability of reliable algorithms for their computation. It is obvious that there is no an universal procedure to compute the functional for the case of general time-delay systems, since we have no such a procedure even for the case of delay free ones. But for linear system such procedures exist both for the delay free and time-delay systems.

In the case of a linear delay free system the computation of quadratic Lyapunov functions is reduced to the solution of the corresponding Lyapunov matrix equation. It is important to underline that here the computation started with the selection of an appropriate quadratic form not for the Lyapunov function but for the time derivative and then the corresponding Lyapunov matrix for the Lyapunov function itself is computed [1].

In the case of linear time-delay systems quite frequently a different scheme is used first a positive definite Lyapunov-Krasovskii functional with some free parameters is selected and then the parameters are used to ensure that the time derivative of the functional along the solution of the system becomes negative. Usually, conditions obtained with this scheme are expressed in the form of linear matrix inequalities (LMIs), [35]. On the one hand, this scheme provides easily verified stability conditions. On the other hand, one does not know beforehand whether or not a time-delay system under study satisfies these conditions. This means that one may obtain in this way a variety of stability conditions but no one of them suits a given exponentially stable time-delay system.

In the seminal paper by Y.M. Repin [34] the computation of functionals with a prescribed time derivative has been initiated. There a general quadratic functional has been differentiated along the solution of a linear system with one time delay and then the derivative has been equated to another quadratic functional. This gave rise to a set of equation for matrices that defined the original functional. The set includes partial differential equations, ordinary differential equations and algebraic equations. If one omits certain terms in the quadratic Lyapunov functionals then is possible to exclude partial differential equations from the set. Later the equations have been subjected to thorough analysis in [2], [3], [4], [5], [6], [14], [13]. It has been clarified that a Lyapunov-Krasovskii functionals obtained in this way are defined by special matrix valued functions of a scalar variable. The functions are known now as delay Lyapunov matrices. They play for the time-delay systems the same role that classical Lyapunov matrices do in the case of linear delay free systems. In [12] a necessary and sufficient condition for the set of equation admits a solution has been derived, see also [19]. This condition is a natural extension of the classical Lyapunov condition known for the Lyapunov matrix equations. In [23] it has been demonstrated that adding to the functionals some new terms allows to prove that the modified functionals admit both lower and upper quadratic estimates of the form needed in the Krasovskii Theorem. The modified functionals are known as the complete type ones. They may be used not only to check stability of a system but also for robustness analysis and the computation of exponential bounds for the system solutions [20], [23], .

An interesting research topic is to express stability conditions of a time delay system in terms of delay Lyapunov matrices. Recently several important results on this direction have been reported in literature [31], [9], [8], [11].

An important issue in practical applications of the complete type functionals is the computation of delay Lyapunov matrices. It has been shown that the matrices depends continuously on the system delays and on the right hand side of the time-delay system [7]. In spite of the fact that these results have been derived for exponentially stable time-delay systems there is a strong feeling that they hold for the general case, as well. For some classes of time-delay systems there are efficient numerical schemes [16]-[18], but we are still far from the solution of the problem in general. Even in reasonably simple cases the computation becomes difficult due to very high dimension of systems to be solved. This means that new numerical algorithms capable to deal with high dimensional systems are needed.

A very promising line of research has been started in where it has been shown that a combination of Lyapunov-Krasovskii functional approach with basic ideas of Razumikhin approach [32], [33], provides new unexpected results, [29], [30].

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