Single-Delay Proportional-Retarded (PR) Protocols for Fast Consensus in a Multi-Agent System

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Abstract: This paper investigates a derivative-free control scheme called the single-delay Proportional-Retarded (PR) protocol to achieve fast consensus in a multi-agent system (MAS) with double-integrator agent dynamics. The PR protocol intentionally introduces a delay in the feedback loop to create a derivative-like mechanism that relies only on position measurements, thus providing high reactivity on the system while mitigating undesirable noise effects. The main result shows how the PR parameters must be tuned, subject to the MAS topology, to effectively place the spectral abscissa of the collective dynamics at a desired locus with the purpose of achieving fast consensus.

Keywords: Multi-agent systems, fast consensus, delay-based control, pole placement

1. INTRODUCTION

Deliberate introduction of delays in a controller to achieve stability and/or attain certain performance in closed-loop settings has been long known (Suh and Bien, 1979; Pyragas, 1992; Kokame et al., 2001). Such controllers, also known as *delay-based controllers*, have various forms (Abdallah et al., 1993; Atay, 1999; Ulsoy, 2015). A largely studied one utilizes time differencing, in place of a derivative controller, to create derivative-like effects while avoiding noise amplification and additional filtering considerations (Fridman and Shaikhet, 2016; Özbek and Eker, 2017a,b; Ramírez, 2015; Ramírez et al., 2013).

The controller utilizing the time differencing idea is called the *retarded controller* as it naturally uses a delay term for the time differencing operation. These controllers have been combined with standard proportional and integral controllers, yielding PR, IR and PIR controllers, to achieve certain stability and performance characteristics in closed loop settings (Hernández-Díez et al., 2017; Ramírez et al., 2015, 2016b; Ramírez, 2015).

Simple implementations of PR and PIR controllers make them attractive and desirable alternatives over standard PD and PID controllers. Indeed, in single-input single-output (SISO) systems, superiority of PR and PIR in terms of noise attenuation have already been demonstrated (Ramírez et al., 2016a; Özbek and Eker, 2017a,b). At this point, one wonders how and in what ways such controllers could also be implemented in network settings, for example, in multi-agent systems (MAS).

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One challenge in designing PR controllers for MAS is that the design problem becomes a large scale one. A remedy to this issue is to establish certain decomposition properties of the corresponding eigenvalue problem, mainly, by decomposing the entire system into subsystems and separately treating the design of each subsystem, combination of which represents the dynamics of the MAS. Although such a decomposition idea is beneficial, the main challenge in this design problem is that the PR controller designed based on one of the subsystems may not be necessarily an ideal one for the remaining subsystems. That is, subsystems *compete* against each other.

To address the above described design problem, one has two options: (a) a multiple-delay PR controller with heterogeneous gains (Ramírez and Sipahi, 2018b), (b) a single-delay PR controller (Ramírez and Sipahi, 2018a). In (a), one completely decouples the subsystems; hence each subsystem is independently designed without any competition. But, this controller is quite complex with multiple gains and delays. In (b), the controller has much simpler form; it has three parameters to tune, which is attractive, but the PR design based on one subsystem may not be adequate for the remaining subsystems.

Simplicity of the single-delay PR controller is attractive however the arising competition between subsystems of the MAS must be understood in order to better harvest the capabilities of PR in network settings; an open problem to the best of our knowledge. To this end, here we start with a single-delay PR controller implemented for each double-integrator agent dynamics of an MAS with undirected graph topology. We next examine the design of PR controller for various subsystems of the MAS and investigate the performance degradation in the remaining subsystems. This degradation, as we show, is determined by the *ratios* between pairs of MAS graph Laplacian eigenvalues. We therefore utilize this knowledge to propose the parameter settings to be used to design the PR controller in a way that subsystem competition does not influence the goal to achieve a desired performance from the MAS.

2. PRELIMINARIES

Before presenting the theoretical framework, we first review the network system under study and a standard modal decomposition of this dynamics.

2.1 Problem formulation

We consider a system with n identical agents whose dynamics is given by the double integrator plant [1]

$$\dot{p}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), \tag{1}$$

where p_i, v_i, and u_i are respectively the position, velocity, and control input of the ith agent. Notice that (1) can be constructed with either homogeneous or heterogeneous agents, simply by scaling each u_i by a constant α_i.

The network in (1) is described with an undirected graph $\mathcal{G} = (N, E)$ where $N = \overline{1, n} \equiv \{1, \ldots, n\}$ is the set of nodes and $E \subset N \times N$ is the set of edges. Each edge has a weight $a_{ij} = a_{ji} > 0, i \neq j$, where the edge $(i, j) \in E$ indicates that agent i receives information from agent j if $a_{ij} \neq 0$. As per the consensus nature, the Laplacian matrix $\mathbf{L} = [-a_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{G}$ has zero row sum; $a_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$. Assuming the agents are connected, $\mathbf{L}$ has one zero eigenvalue and its remaining eigenvalues are real (Horn and Johnson, 1988) and positive (Olfati-Saber and Murray, 2004). Hereafter, we assume that $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n$ holds noting also that the developments can easily be extended to the case of repeated eigenvalues (see, e.g., Section 4).

To achieve agreement in position and velocity amongst all the agents, motivated by successful implementation of the PR protocol originally developed for SISO systems (Suh and Bien, 1979; López et al., 2017), here we start with the same protocol and implement it for each of the agents, subject to network topology,

$$u_i(t) = k_p \sum_{j=1}^{n} a_{ij} \Delta p_{ji}(t) - k_r \sum_{j=1}^{n} a_{ij} \Delta p_{ji}(t - h) \tag{2}$$

where $\Delta p_{ji}(t) = p_j(t) - p_i(t)$, k_p and k_r globally modulate the strength of the proportional and retarded actions, and $h \geq 0$ is an intentional delay induced in the input to an agent. The main motivation to introduce (2) here is that the presence of delay can help improve dynamic response as already shown in (Ramírez et al., 2016a; Suh and Bien, 1979; Ulsoy, 2015) where an intentional delay is used to mimic a realizable derivative effect and to enhance performance. Moreover, in many cases, only p_i is measurable, and differentiating it to estimate v_i can be prohibitive especially in the presence of noise (Ramírez et al., 2015). In view of this rationale, in noisy environments, the PR protocol can be considered more preferable over traditional PD protocols. Motivated by these observations, here we aim to investigate the opportunity of using delay as part of protocol (2) for the multi-agent system (1). We say that (2) solves the consensus problem if $\lim_{t\to\infty} \|p_i(t) - p_j(t)\| = 0$ and $\lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0$ for all $i, j \in N$.

To address the above stated problem, we start with some definitions. Let $\mathbf{x} = \left(p_1, v_1, \cdots p_n, v_n\right)^\top$ be the stack vector of the states at all nodes, then the matrix form of (1) with (2) is conveniently written as

$$\dot{\mathbf{x}}(t) = \left(\mathbf{I}_n \otimes \mathbf{J} - \mathbf{L} \otimes \mathbf{K}_p\right) \mathbf{x}(t) + \mathbf{L} \otimes \mathbf{K}_r \mathbf{x}(t - h), \tag{3}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix and

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \mathbf{K}_p = \begin{pmatrix} 0 & 0 \\ k_p & 0 \end{pmatrix}, \ \mathbf{K}_r = \begin{pmatrix} 0 & 0 \\ k_r & 0 \end{pmatrix}. \tag{4}$$

Stability of (3) is important and can be studied through the roots of its characteristic equation given by

$$f(s) = \det\left[s\mathbf{I}_{2n} - \left(\mathbf{I}_n \otimes \mathbf{J} - \mathbf{L} \otimes \mathbf{K}_p\right) - \mathbf{L} \otimes \mathbf{K}_r e^{-hs}\right] = 0. \tag{5}$$

However, here we are interested with the exponential decay rate of the consensus dynamics with a given degree γ, namely, with the γ-stability of (3), which is directly associated with fast/slow consensus reaching. The main problem of this article is then: find the analytical formula relating (h, k_p, k_r) that creates the maximum exponential decay rate for γ based on the roots of (5).

It is well known that the decay rate of (3) is associated with the distribution of the roots of (5). In particular, the above problem is better understood as the problem of minimizing the spectral abscissa γ^* of (3), (Hale and Verduyn Lunel, 1993, Theorem 6.2). More precisely, how should the parameters h, k_p and k_r be selected such that real part of the rightmost root of (5) is pushed into the left hand-side of the complex plane as far as possible?

2.2 Modal decomposition of the consensus dynamics

Finding an analytical relation between (h, k_p, k_r) and γ-stability is not an easy task. In fact, analysis of γ-stability over (5) is prohibitive, especially for large scale problems. We summarize below an approach that can address this problem, see the main results in (Ramírez and Sipahi, 2018b).

Proposition 1. The characteristic equation (5) satisfies

$$f(s) = \prod_{m=1}^{n} f_m(s) = \prod_{m=1}^{n} \left(s^2 + \lambda_m k_p - \lambda_m k_r e^{-sh}\right), \tag{6}$$

where λ_m is an eigenvalue of $\mathbf{L}$.

Proof. Since the graph is undirected, $\mathbf{L}$ is symmetric, hence the Schur's theorem (Horn and Johnson, 1988) guarantees the existence of a nonsingular orthogonal matrix $\mathbf{U} \in \mathbb{R}^{n\times n}$, such that $\mathbf{L} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$ holds, where $\mathbf{D}$ is a diagonal matrix formed with the eigenvalues of $\mathbf{L}$. Under this unitary transformation, introduce the change of variable $\mathbf{x}(t) = \mathbf{U}\boldsymbol{\xi}(t)$, which transforms system (3) into

$$\dot{\boldsymbol{\xi}}(t) = \left(\mathbf{I}_n \otimes \mathbf{J} - \mathbf{D} \otimes \mathbf{K}_p\right) \boldsymbol{\xi}(t) + \mathbf{D} \otimes \mathbf{K}_r \boldsymbol{\xi}(t - h). \tag{7}$$

The fact that (7) is in diagonal form implies that (3) can be treated as a set of n decoupled subsystems with dynamics

$$\dot{\boldsymbol{\xi}}_m(t) = \left(\mathbf{J} - \lambda_m \mathbf{K}_p\right) \boldsymbol{\xi}_m(t) + \lambda_m \mathbf{K}_r \boldsymbol{\xi}_m(t - h). \tag{8}$$

The characteristic equation of (8) is $f_m(s) = s^2 + \lambda_m k_p - \lambda_m k_r e^{-sh}$, which is a factor of $f(s)$ in (5). □

[1] These types of dynamics attract strong interest both in engineering and physics literature as they enable studying fundamental benchmark problems, see (Ramírez and Sipahi, 2018b) for a thorough list of references.

Remark 2. Notice that $\lambda_1 = 0$ corresponds to the consensus state $s=0$. Hereafter, the case $m = 1$ is ignored in the stability analysis since this subsystem corresponds to the consensus state, independent of delay h. We shall say that "consensus γ-stability" holds if and only if all s satisfying $f_m = 0$, $m = \overline{2, n}$ have negative real parts less than γ.

With Proposition 1 at hand, and keeping Remark 2 in mind, we now are able to separately analyze the γ_m-stability of the subsystems in (8) and use this analysis to conclude about the γ-stability of the complete system in connection with the design of (h, k_p, k_r). To this end, let γ_m^* be the spectral abscissa of (8) and define

$$\gamma^* = \max_{2 \le m \le n} \{\gamma_m^*\}. \tag{9}$$

It follows that $\gamma^* < 0$ holds if and only if $\gamma_m^* < 0$ for all possible m. One important point here is that h, k_p and k_r are distributed throughout all the factors in (6). That is, the design of (h, k_p, k_r) using only one factor associated with one subsystem may produce favorable results for this subsystem, but not necessarily for all subsystems. Therefore, it is necessary to study how this choice impacts the stability of the rest of the factors.

2.3 Tuning of the PR protocol for a stable subsystem

We first investigate a generic factor f_q and show how its spectral abscissa γ_q^* can be placed at any desired position γ_{d_q} thus guaranteeing γ_{d_q}-stability provided that $\gamma_{d_q} < 0$. To this end, let us introduce the change of variable $s \to (s+\gamma_q)$ by which the real part of the Laplace operator is shifted by γ_q. From (6), the change of variable yields $f_q(s+\gamma_q) = (s+\gamma_q)^2 + \lambda_q k_p - \lambda_q k_r e^{-(s+\gamma_q)h}$. The purpose here is to push the roots of f_q as deep as possible into the left-half of the complex plane using the shift γ_q. To this end, a minimum value for γ_q^* is to be found in terms of k_p and further associated with the design parameters in (h, k_r) domain. For conciseness, we summarize from (Ramírez and Sipahi, 2018b) the following proposition, see also (Ramírez et al., 2015) for SISO implementations:

Proposition 3. (Ramírez and Sipahi (2018b)). Let $k_p > 0$ and $\lambda_q > 0$ be given, then γ_q^* exhibits a minimum in (h, k_r) domain at

$$\gamma_q^* = -\sqrt{\lambda_q k_p}, \tag{10}$$

subject to

$$(h, k_r) = \left(-\frac{1}{\gamma_q^*}, -\frac{2\gamma_q^*}{\lambda_q h e^{-\gamma_q^* h}} \right). \quad \square \tag{11}$$

The above result reveals an exact minimum for γ_q^*. Furthermore, this characterization may be used to achieve a desired exponential decay rate γ_{d_q} for one of the subsystems in (8), ultimately establishing its γ_{d_q}-stability, via the analytical tuning of h, k_p and k_r as stated next.

Proposition 4. Let $\gamma_{d_q} < 0$ be a desired exponential decay rate, then the spectral abscissa of the qth subsystem in (8) is placed at γ_{d_q} by tuning the gains of the PR protocol as

$$(h, k_p, k_r) = \left(-\frac{1}{\gamma_{d_q}}, \frac{\gamma_{d_q}^2}{\lambda_q}, \frac{2\gamma_{d_q}^2 e^{-1}}{\lambda_q} \right). \tag{12}$$

Proof. The result follows from Proposition 3 by defining $\gamma_q^* \equiv \gamma_{d_q}$. Then, solving k_p from (10) and algebraic manipulation of (11) yield (12). $\square$

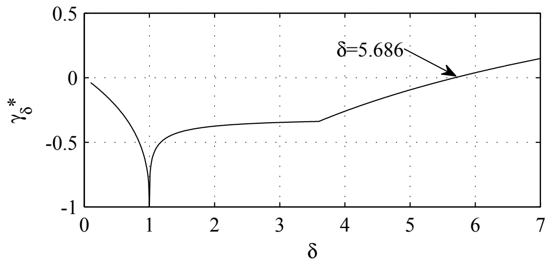

Fig. 1. Non-dimensional spectral abscissa γ_δ^* with respect to δ. Computation using TRACE-DDE (Breda et al., 2009) and (14).

As mentioned above, the PR design based on one subsystem should not be detrimental for the other subsystems to collectively achieve the control design goal at hand. Based on Proposition 4, we next investigate the impact of λ_m on the stability of these subsystems.

3. STABILITY ANALYSIS

The analytical tuning in the previous section is now employed to study the stability of the overall system in connection with the Laplacian eigenvalues associated with the topology of the network.

3.1 Non-dimensionalization

Motivated by (Zítek et al., 2013), the following non-dimensionalization is performed to facilitate the stability analysis of (3) subject to the tuning of the PR protocol in (12). To this end, let the Laplace operator s and the mth factor in (6) be scaled by $-\gamma_{d_q}$ and $1/\gamma_{d_q}^2$, respectively,

$$\frac{f_m(-s\gamma_{d_q})}{\gamma_{d_q}^2} = \frac{s^2\gamma_{d_q}^2}{\gamma_{d_q}^2} + \frac{\lambda_m k_p}{\gamma_{d_q}^2} - \frac{\lambda_m k_r e^{s\gamma_{d_q} h}}{\gamma_{d_q}^2}. \tag{13}$$

Substituting (h, k_p, k_r) in (12) into the above equation and defining $\delta \equiv \lambda_m/\lambda_q$ and $f_\delta(s) \equiv f_m(-s\gamma_{d_q})/\gamma_{d_q}^2$, the tuned characteristic factors take the final form

$$f_\delta(s) = s^2 + \delta - 2\delta e^{-(s+1)}, \tag{14}$$

where in this case γ_δ^* is defined as the real part of the rightmost root of (14), then the linear function

$$\gamma_m^* = -\gamma_{d_q}\gamma_\delta^* = -\gamma_{d_q}\gamma_{\lambda_m/\lambda_q}^*, \tag{15}$$

maps any γ_δ^* value to the original coordinates γ_m^*. Using next TRACE-DDE (Breda et al., 2009), we record in Fig. 1 the behavior of γ_δ^* as δ increases. Observe that γ_δ^* attains its minimum at -1 for $\delta = \lambda_m/\lambda_q = 1$, which holds only if $q = m$. Then, substituting $\gamma_\delta^* = -1$ into (15) we readily obtain that $\gamma_m^* = \gamma_{d_m}$. In other words, the spectral abscissa of the mth subsystem is placed at the desired locus γ_{d_m}, which is consistent with the design presented in Proposition 4. Since γ_δ^* is minimal at $\delta = 1$, it follows that $\delta \neq 1$ always creates larger γ_δ^* values. In other words, the analytical placement of γ_m^* shifts the spectral abscissas of the rest of the subsystems towards right on the complex plane. Furthermore, stability is preserved only in the interval $0 < \delta < \bar{\delta}$, where the upper-bound is easily found by crossing frequency analysis of (14), $\bar{\delta} = \pi^2/(1 - 2\, e^{-(j\pi+1)}) \approx 5.686$, see (Michiels and Niculescu, 2007).

Proposition 5. The consensus dynamics (3), subject to (h, k_p, k_r) in (12) with $q \in \overline{2, n}$, is stable if and only if $0 < \lambda_m/\lambda_q < \bar{\delta}$ holds for all $m = \overline{2, n}$. $\square$

At this point, the design of the PR controller is able to provide stability. Recall however that the main objective here is to achieve fast consensus, which is related with the placement of the spectral abscissa of the overall system. In what follows, drawing on the above developments, we perform a simple re-design of the proposed protocol with which fast consensus can be achieved.

3.2 Tuning of the PR protocol for rightmost root placement

To render the network with improved settling times, we rely on the previous analysis to place the spectral abscissa of the consensus dynamics at a desired locus. From the above discussions, we can conclude from (9) and (15) that

$$\gamma^* = \max_{2\le m\le n}\{\gamma_m^*\} = -\gamma_{d_q}\max_{\delta>0}\{\gamma_\delta^*\} = -\gamma_{d_q}\bar{\gamma}_\delta^*, \quad (16)$$

for some $q \in \overline{2,n}$. With the above in mind, the following proposition places the spectral abscissa of the entire network at a desired locus.

Proposition 6. Let $q \in \overline{2,n}$ be fixed and a desired spectral abscissa $\gamma_d < 0$ be given. Then, for network (3), at least one dominant root at γ_d is placed by tuning the gains of the PR protocol as

$$(h, k_p, k_r) = \left(\frac{\bar{\gamma}_\delta^*}{\gamma_d}, \frac{\gamma_d^2}{\bar{\gamma}_\delta^{*2}\lambda_q}, \frac{2\gamma_d^2 e^{-1}}{\bar{\gamma}_\delta^{*2}\lambda_q}\right), \quad (17)$$

where $\bar{\gamma}_\delta^* = \max_{\delta>0}\{\gamma_\delta^*\}$ is obtained from $f_\delta(s) = s^2 + \delta - 2\delta e^{-(s+1)}$ as the maximum value that the real part of its rightmost root can exhibit, subject to $\mathcal{G}$, with $\delta = \lambda_m/\lambda_q$, $m = \overline{2,n}$ for some fixed $q \in \overline{2,n}$.

Proof. The proof follows by noting that $\gamma^* = -\gamma_{d_q}\bar{\gamma}_\delta^*$. Hence, choosing $\gamma_{d_q} = -\gamma_d/\bar{\gamma}_\delta^*$ yields $\gamma^* = \gamma_d$. Further substitution of γ_{d_q} into (12) yields (17). □

The above proof completes the main contributions of this study. Specifically, the PR controller enables a desired spectrum optimization that ultimately ensures a dominant desired spectral abscissa. This becomes possible by optimizing the spectrum based on the worst case subsystem and over-designing the remaining subsystems.

4. CASE STUDIES

We now verify the theoretical results via numerical examples under different topologies. Since the developments are fully analytical, the main result in Proposition 6 must guarantee the exact placement of the spectral abscissa. To conserve space, some position plots are suppressed. In all the cases, a five-agent MAS is considered, $n = 5$. For simplicity, agents' coupling strengths are taken to be homogeneous $a_{ij} = a_{ji} = a = 0.2$, subject to $\mathcal{G}$. Initial positions $[0.180, 0.231, 0.088, 0.148, 0.162]^\top$ and velocities $[0.131, 0.356, 0.059, 0.159, 0.254]^\top$ at $t = 0$ are randomly picked for all the cases. To emulate the availability of the initial conditions in $-h \le t \le 0$, in the practical implementation of the PR protocol, we let $u_i(t) = 0$ for $t \in [0, h] \equiv t_h$. After t_h has elapsed, the PR protocol is activated and hence, the stability features of the collective dynamics (3) are properly captured by (6). Time simulations are conducted using MATLAB/ODE4 with a fixed step size 0.001 sec, and the spectrum of the dynamics is computed using QPmR (Vyhlídal and Zítek, 2009).

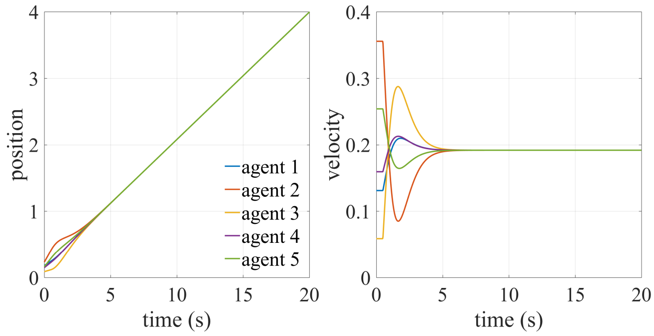

Fig. 2. Five agents under a complete graph with $\gamma_{d_q} = -2$

4.1 Complete graph as ideal

Consider the complete graph with PR and let $\gamma_d = -2$. The non-zero eigenvalues of $\mathbf{L}$ are $\lambda_2 = \cdots = \lambda_n = na$ (Koh and Sipahi, 2016). That is, there exist $n-1$ identical subsystems. This case study is considered *ideal* since the ratio between any pairs of eigenvalues, δ, can only be unity, and consequently no competition among subsystems. That is, the PR design based on one subsystem will place all the subsystem spectral abscissas at $\gamma_{d_q} = \gamma_d = -2$. In view of this, from (12), the PR controller parameters are obtained $(h, k_p, k_r) = (0.5, 4, 2.943)$. Fig. 2 presents the positions and the velocities of the agents, each equipped with the designed PR controller.

4.2 Undirected line topology

Consider now an undirected line graph. The eigenvalues of $\mathbf{L}$ are $\left\{0, \frac{3-\sqrt{5}}{2}a, \frac{5-\sqrt{5}}{2}a, \frac{3+\sqrt{5}}{2}a, \frac{5+\sqrt{5}}{2}a\right\}$. By inspection of γ_δ^* in Fig. 1 subject to δ, we determine λ_q minimizing γ^*. Table 1 is provided for this purpose, which suggests two messages. Firstly, since $\lambda_5/\lambda_2 = 9.472 > 5.686$, this leads to positive γ_5^*, that is, instability. Hence, $\lambda_q \ne \lambda_2$. Secondly, in terms of attaining the convergence rate γ^*, one should select $\lambda_q = \lambda_3$, see Table 1. Now considering the scaling property in (15) and doubling γ_{d_q} to $\gamma_{d_q} = -2$ for consistency with the other cases, the numerical values of γ_m^* in Table 1 will also double. Indeed, these observations are also validated with computations, see Fig. 3.

Table 1. Spectral abscissa values γ_m^* obtained with the non-dimensional system for $\delta = \lambda_m/\lambda_q$ for $q = \overline{2,5}$

	$\delta = \lambda_m/\lambda_q$	γ_m^*
$\lambda_q = \lambda_5$	[0.106,0.382,0.724,1]	[-0.040,-0.163,-0.380,-1]
$\lambda_q = \lambda_4$	[0.146,0.528,1,1.382]	[-0.056,-0.243,-1,-0.435]
$\lambda_q = \lambda_3$	[0.276,1,1.894,2.618]	[-0.112,-1,-0.380,-0.353]
$\lambda_q = \lambda_2$	[1,3.618,6.854,9.472]	[-1,-0.334, 0.134, 0.353]

4.3 Undirected star topology

In this case, the eigenvalues of $\mathbf{L}$ are given by $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = \lambda_4 = a$ and $\lambda_5 = na$, hence one has two different choices for λ_q to be used in tuning the PR controller based on (12). For each selection, the set of δ becomes either $\{1, 1, 1, 5\}$ for $\lambda_q = \lambda_2$ or $\{0.2, 0.2, 0.2, 1\}$ for $\lambda_q = \lambda_5$. From Fig. 1, based on δ, the

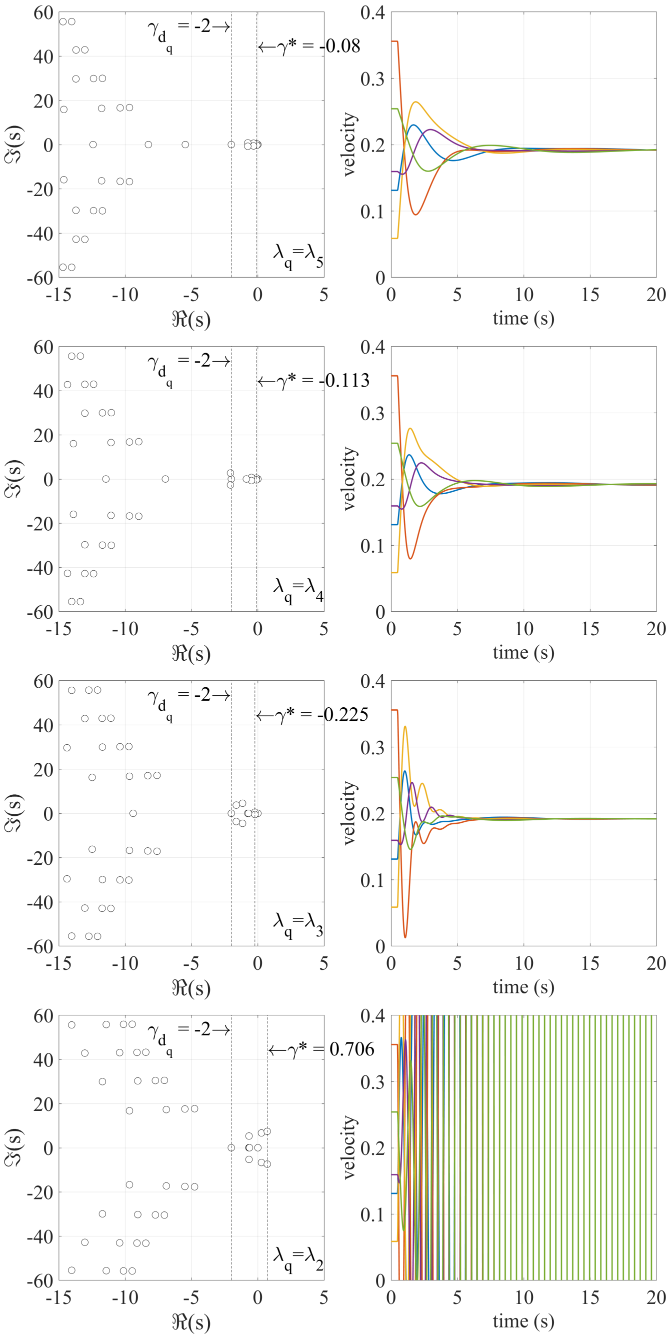

Fig. 3. Five agents under an undirected line topology with $\gamma_{d_q} = -2$ and $h = 1/2$; $(k_p, k_r) = (5.528, 4.067)$ for $\lambda_q = \lambda_5$; $(k_p, k_r) = (7.6393, 5.6207)$ for $\lambda_q = \lambda_4$; $(k_p, k_r) = (14.472, 10.648)$ for $\lambda_q = \lambda_3$; $(k_p, k_r) = (52.361, 38.525)$ for $\lambda_q = \lambda_2$ using (12).

set of γ_m^* is given respectively by $\{-1, -1, -1, -0.0938\}$ and $\{-0.0789, -0.0789, -0.0789, -1\}$. Therefore, we expect that the selection $\lambda_q = \lambda_2$ achieves a faster convergence rate since $-0.0938 < -0.0789$. Next, doubling to $\gamma_{d_q} = -2$ for consistency with the other cases, these observations are validated in computations, see Fig. 4.

4.4 Undirected ring topology

Consider an undirected ring topology. The eigenvalues of $\mathbf{L}$ are $\left\{0, \frac{5-\sqrt{5}}{2}a, \frac{5-\sqrt{5}}{2}a, \frac{5+\sqrt{5}}{2}a, \frac{5+\sqrt{5}}{2}a\right\}$. In this case, one has two options in terms of picking λ_q for tuning (h, k_p, k_r):

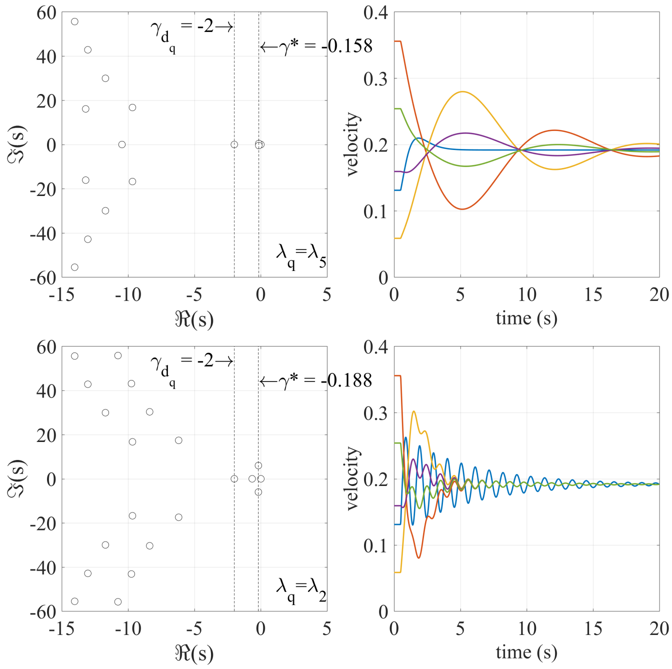

Fig. 4. Five agents under an undirected star topology with $\gamma_{d_q} = -2$ and $h = 1/2$. (Top) $(k_p, k_r) = (4, 2.9430)$ for $\lambda_q = \lambda_5$. (Bottom) $(k_p, k_r) = (20, 14.715)$ for $\lambda_q = \lambda_2$ using (12).

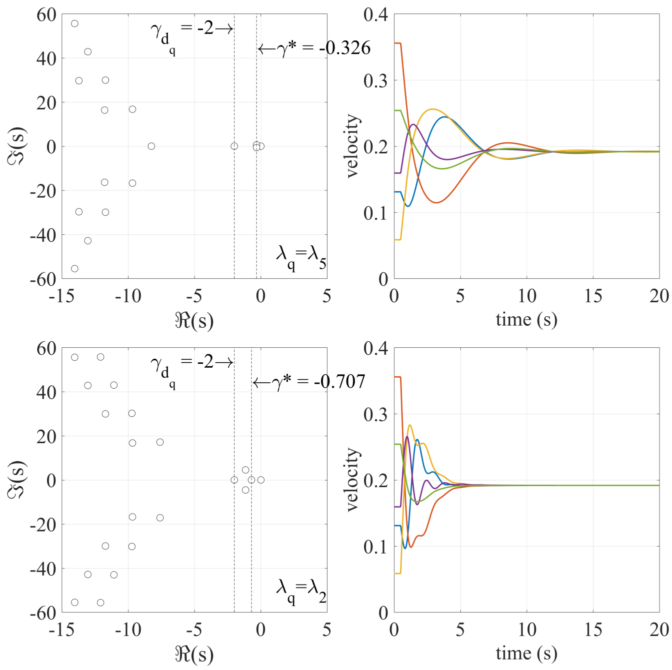

Fig. 5. Five agents under an undirected ring topology with $\gamma_{d_q} = -2$ and $h = 1/2$; (k_p, k_r) computed with (12).

(i) If $\lambda_q = \lambda_5 = \lambda_4 = \frac{5+\sqrt{5}}{2}a$, the set of δ ratios is $\{0.382, 0.382, 1, 1\}$. With this, for $\gamma_{d_q} = -1$, the set of γ_m^* is obtained as $\{-0.163, -0.163, -1, -1\}$ from Fig. 1. Based on the scaling in (15), it is easy to see that the set of γ_m^* reads $\{-0.326, -0.326, -2, -2\}$ when γ_{d_q} is doubled to $\gamma_{d_q} = -2$. In this case, PR parameters read $(h, k_p, k_r) = (0.5, 5.528, 4.067)$.

(ii) If $\lambda_q = \lambda_3 = \lambda_2 = \frac{5-\sqrt{5}}{2}a$, the set of δ ratios becomes $\{1, 1, 2.618, 2.618\}$. With this, for $\gamma_{d_q} = -1$, the

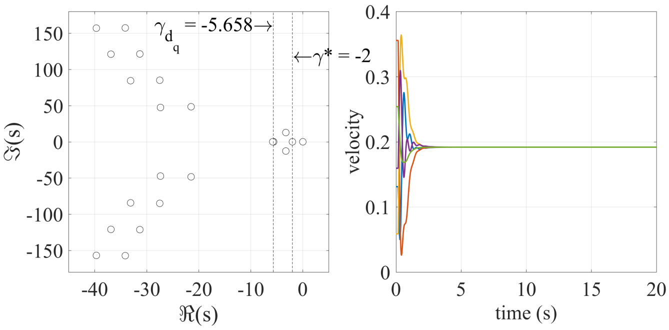

Fig. 6. Five agents under an undirected ring topology with $\gamma^* = -2$ and (h, k_p, k_r) computed with (17).

set of γ_m^* is obtained as $\{-1, -1, -0.3534, -0.3534\}$ from Fig. 1. Similarly, for $\gamma_{d_q} = -2$, the entries in the set of γ_m^* are doubled to $\{-2, -2, -0.7068, -0.7068\}$ as per the scaling property in (15). In this case, PR parameters read $(h, k_p, k_r) = (0.5, 14.472, 10.648)$.

By inspecting $\gamma^* = \max_{2\leq m\leq 5}\{\gamma_m^*\}$ between (i) and (ii), we have that MAS with PR controller in (ii) achieves faster consensus than in (i), owing to its smaller γ^*, i.e., $-0.707 < -0.326$. This conclusion is consistent with time simulations (Fig. 5). Moreover, γ^* of MAS can be attained by designing the PR controller based on (17). Given $\gamma_d = \gamma^* = -2$ and that we have $\lambda_q = \lambda_2$ and $\bar{\gamma}_\delta^* = -0.354$ (see (ii)), the PR parameters now read $(h, k_p, k_r) = (0.1767, 115.8776, 85.2579)$. Then the new set of γ_m^* becomes $\{-5.658, -5.658, -2, -2\}$, and as expected $\gamma^* = -2$. Simulation results are shown in Fig. 6.

5. CONCLUSIONS

This paper studies the convergence rate of an LTI consensus dynamics using a Proportional Retarded (PR) protocol in a single-delay setup. To facilitate the implementation of the proposed protocol, we use spectral analysis to optimize the dominant modes of the system resulting in algebraic tuning formulas for the parameters of the controller, ultimately placing the spectral abscissa at a desired position thus ensuring fast settling times. The proposed approach is scalable and computationally amenable for practitioners. Moreover, the proposed approach can be applied regardless of the size and the type of network, heterogeneous/homogeneous coupling strength as long as the underlying graph of the dynamics is undirected.

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