Design of PID Controllers under Saturation and Time Delay

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The aim of this abstract is to show a robust PID design where the gain margin is preserved under saturation. We use results presented in [2] and [4] to the design of PID controllers by means of the convex-concave procedure (CCP) [1, 3].

The design of the PID controller is based on maximising its low-frequency gain subject to predefined stability margins as in [1]. Additional constraints are imposed to guarantee the closed loop stability in presence of saturation and delay. These requirements are taken from [4] and [2].

The proposed methodology is available for any plant and any controller structure with affine parameters. In this manuscript, we consider the closed-loop system composed by the following plant and controller as an example:

- plant $P(s) = \frac{ke^{-sh}}{\tau s+1}$ with k, h, $\tau \geq 0$;
- PID controller $C(s) = k_p + \frac{k_i}{s} + k_d s$.

Let $G(s) = C(s)P(s)$, and $x = \begin{bmatrix} k_p & k_i & k_d \end{bmatrix}^T$.

In [4], the stability of the closed-loop interconnection between $G \in \mathbf{RH}_\infty$ with delay and saturation can be guaranteed by conditions on the linear system $G(j\omega)$, where the Off-Axis Circle Criterion (OACC) can be used to show that the plant satisfies the Kalman Conjecture. Furthermore, it can be proved that the same conclusion is valid when an integrator is introduced, if $\lim_{s\to 0} sG(s) > 0$ [2]. Then, the conditions are represented below.

$$\Im\left(\overline{G_\omega} G_{\omega\omega}\right) = \frac{k^2(x^T Q_1 x)}{\omega^2(\tau^2\omega^2+1)^3} \leq 0, \tag{1}$$

$$\frac{|G(j\omega)|}{d\omega} = \frac{k(x^T Q_2 x)}{\omega^2(\tau^2\omega^2+1)^{\frac{3}{2}}\sqrt{k_d^2\omega^4 - 2k_d k_i \omega^2 + k_i^2 + k_p^2\omega^2}} \leq 0, \tag{2}$$

where Q_1 and Q_2 can be obtained by straightforward calculation.

These two conditions are equivalent to

$$x^T Q_i x \leq 0 \quad (i = 1, 2). \tag{3}$$

In general, both conditions are non-convex functions, but they can be rewritten as a difference of convex functions.

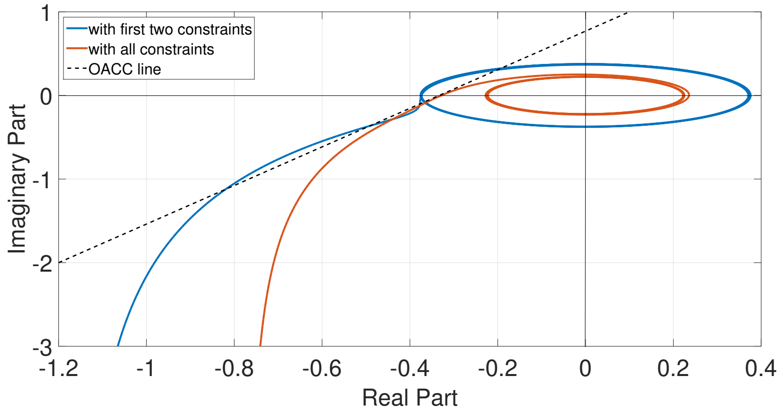

Figure 1: Nyquist Plots under Different Constraints

Given a symmetric matrix Q with positive and negative eigenvalues, its LDU factorisation satisfies $U = L^T$, and D is a diagonal matrix with both positive and negative eigenvalues. Therefore,

$$Q = L(D_p + D_n)L^T, \tag{4}$$

where D is decomposed into a diagonal matrix with positive values D_p and the other one with negative values D_n. In this way, Q is separated into a positive semidefinite matrix $Q_p = LD_pL^T$ and a negative semidefinite matrix $Q_n = LD_nL^T$. This decomposition serves to find convex approximation of the non-convex conditions presented before.

A linear approximation with respect to a point x_k of the non-convex term x^TQ_nx is used below to obtain a convex expression.

$$x^TQ_px + x^TQ_nx \leq x^TQ_px + 2x_k^TQ_nx - x_k^TQ_nx_k \leq 0. \tag{5}$$

This convex approximation is used in order to apply the CCP to the following non-convex optimisation problem.

Let

$$\begin{aligned} \underset{x}{\text{minimise}} \quad & -k_i \\ \text{subject to} \quad & ||S||_\infty \leq M_s; \\ & ||T||_\infty \leq M_t; \\ & \Im\left(\overline{G_\omega} G_{\omega\omega}\right) \leq 0; \\ & \frac{|G(j\omega)|}{d\omega} \leq 0. \end{aligned} \tag{6}$$

See [1] for the motivation of the previous problem and a brief exposition of the CCP.

Let $k = \tau = h = 1$, and $M_s = M_t = 1.6$. The figure below shows the Nyquist plots of $G(j\omega)$ with PID controllers with different constraints.

In this figure, the blue corresponds to the solution with the first two constraints, and the red corresponds to the solution with all constraints. It is clear the kink in the blue curve is avoided in the red. Moreover, as illustrated by the red curve, the closed-loop system with saturation and delay is stable according to the OACC (dashed line). However, without the two last constraints, the argument is invalid for systems with nonlinearities.

[1] Hast M., Åström K. J., Bernhardsson B., Boyd S. P., PID design by convex-concave optimization, *Proceedings of the 2013 European Control Conference*, 4460–4465, 2013.

[2] Jönsson U. and Megretski A., The Zames-Falb IQC for systems with integrators, *IEEE Transactions on Automatic Control* **45**:560–565, 2000.

[3] Mercader P., Åström K. J., Baños A., Hägglund T., Robust PID design based on QFT and convex-concave optimization, *IEEE Transaction on Control System Technology* **25**(2):441–452, 2017.

[4] Zhang J., Tugal H., Carrasco J., Heath W., Clockwise properties of time delayed systems and their application in absolute stability, *57th IEEE Conference on Decision and Control*, to be submitted, 2018.