Lower Bounds on Delay Margin of Second-Order Unstable Systems ★

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Abstract: This paper examines the delay margin achievable by using PID controllers for linear time-invariant (LTI) systems subject to variable, unknown time delays. We derive explicit lower bounds on the delay margin of second-order unstable delay systems achievable by PID control, which provide *a priori* the ranges of delay over which a second-order delay plant is guaranteed to be stabilizable by a PID controller, and more generally, by a finite-dimensional LTI controller. Analysis shows that our results are less conservative in most cases than lower bounds obtained elsewhere using more sophisticated and general LTI controllers.

Keywords: Delay margin, robust stabilization, uncertain time delay, PID controller.

1. INTRODUCTION

Consider the feedback system depicted in Fig.1, in which $P_\tau(s)$ denotes a plant to be controlled subject to an unknown delay τ, whose transfer function is given by

$$P_\tau(s) = P_0(s)e^{-\tau s}, \quad \tau \geq 0, \tag{1}$$

where $P_0(s)$ is a finite dimensional delay-free plant. The

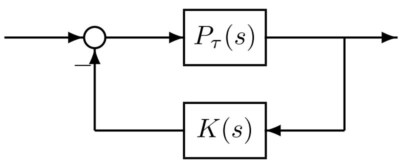

Fig. 1. Feedback control system

problem of *delay margin* is to determine the largest variation range of $\tau \geq 0$ such that $P_\tau(s)$ can be stabilized by a certain finite-dimensional LTI controller $K(s)$ in that range, that is, to compute the delay margin

$$\bar{\tau} = \sup\{\mu \geq 0 : \text{ There exists some } K(s) \text{ that stablizes } P_\tau(s),\ \forall \tau \in [0,\, \mu)\}.$$

A problem of significant practical interest is to find the delay margin achievable by all LTI controllers of the PID structure

$$K_{PID}(s) = k_p + \frac{k_i}{s} + k_d s, \tag{2}$$

in other words, we are interested in

$$\bar{\tau}_{PID} = \sup\{\mu \geq 0 : \text{ There exists some } K_{PID}(s) \text{ that stablizes } P_\tau(s),\ \forall \tau \in [0,\, \mu)\}.$$

The delay margin defined above serves as a fundamental measure of robust stabilization against uncertain time delays which asks the question: in what range of the delay parameter can a system be stabilized by one controller for all possible delays lying within that range? For its fundamental interest, this problem has received considerable attention, we refer to Michels et al. (2002), Miller and Davison (2005), Gaudette and Miller (2016), Bresch-Pietri et al. (2012); Liberis and Krstic (2013), Ju and Zhang (2016); Middleton and Miller (2007); Qi et al. (2017b), Qi et al. (2017a), Michels et al. (2002); Wei and Lin (2017); Yoon and Lin (2013); Zhou et al. (2012) for various methods tackling the problem, which accordingly, result in different bounds on the delay margin depending on the complexities of the control law and control structure. Generally, for high-order systems, the exact delay margin was found only for plants containing one unstable pole and one nonminimum phase zero, achievable by general LTI controllers of a possible high order Middleton and Miller (2007); Qi et al. (2017b).

With a fixed structure and only three parameters available for design, PID control of delay systems poses a more challenging task, and is largely limited to low-order systems. An earlier result on the delay margin achievable by PID control is found in Michels et al. (2002) (see also Michiels and Niculescu (2007) (pp. 154)), where the exact delay margin was obtained for first-order systems controlled by proportional static feedback. With a full PID controller, this margin can be doubled Silva et al. (2002, 2005), which,

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inadvertently, is known to be the maximum delay margin achievable by any finite-dimensional LTI controllers Michels et al. (2002); Middleton and Miller (2007); Qi et al. (2017b). Recently, the authors examined second-order systems Ma and Chen (2017), where a general optimization scheme was developed in search for the delay margin achievable by PID controllers. For second-order anti-stable plants, the problem appears considerably more difficult, and only upper bounds are available.

Our purpose in this note is to derive companion lower bounds on the delay margin of second-order systems achievable by PID control. Results of this nature provide, *a priori*, a range of delay values within which the delay system is guaranteed to be stabilizable by a PID controller, and consequently lower bounds on the delay margin achievable by any finite-dimensional LTI controllers. A comparison to other lower bounds obtained elsewhere, in e.g., Wei and Lin (2017); Zhou et al. (2012), where general LTI controllers are allowed, shows that our bounds can be significantly less conservative. As a final remark, it is known that PID control is generally effective only for systems up to the second-order (see, e.g., Krstic (2017)), and for unstable systems of a higher order, it is likely that PID controllers may not even stabilize the systems free of delay, lest that the systems may contain delays. On the other hand, many industrial processes are often modeled by first- and second-order systems. Under such circumstances, PID control remains to be an attractive means for its low complexity in design and implementation.

2. PRELIMINARY RESULTS

We consider second-order anti-stable delay systems

$$P_\tau(s) = \frac{1}{(s-p_1)(s-p_2)} e^{-\tau s}, \tag{3}$$

where $\mathrm{Re}(p_1) \geq 0$ and $\mathrm{Re}(p_2) \geq 0$. In the sequel, we shall address the cases where p_1 and p_2 are both real unstable poles, or they are an unstable complex conjugate pair. Throughout the paper we shall assume that in the PID controller (2) the integral coefficient $k_i = 0$; that is, only PD controller will be considered. Underlying this consideration is the fact that integral control action will generally reduce the delay margin Ma and Chen (2017).

In conjunction with the delay margin $\bar{\tau}_{PID}$, also of interest is the margin attainable by a particular PID controller with a given set of the controller parameters k_p, k_i, and k_d. This controller-specific delay margin is defined as

$$\bar{\tau}_{PID}(K_{PID}) = \sup\{\mu \geq 0 : K_{PID}(s) \text{ stablizes } P_\tau(s), \\ \forall \tau \in [0, \mu)\}.$$

Clearly,

$$\bar{\tau}_{PID} = \sup\{\bar{\tau}_{PID}(K_{PID}) : \ K_{PID}(s) \text{ stabilizes } P_\tau(s), \\ \forall \tau \in [0, \mu)\}.$$

The delay margin $\bar{\tau}_{PID}$ for second-order delay systems generally presents a difficult problem, which has seldom been addressed in the previous work. Indeed, no explicit, exact delay margin appears to have been found. The following results adopted from Ma and Chen (2017), however, show that it can be determined by finding the maximum of a real function in two real variables, by optimizing $\bar{\tau}_{PID}(K_{PID})$.

Proposition 2.1. Let $P_\tau(s)$ be given by (3), with $p_1 \geq 0$ and $p_2 \geq 0$ being real poles. Suppose that $k_i = 0$. Then for any given $K_{PID}(s)$ to stabilize $P_0(s)$, it is necessary and sufficient that $k_p > p_1 p_2$ and $k_d > p_1 + p_2$. Under these conditions,

$$\bar{\tau}_{PID}(K_{PID}) = \frac{\tan^{-1}\frac{\omega_0}{p_1}}{\omega_0} + \frac{\tan^{-1}\frac{\omega_0}{p_2}}{\omega_0} + \frac{\tan^{-1}\frac{k_d\omega_0}{k_p}}{\omega_0} - \frac{\pi}{\omega_0}, \tag{4}$$

where $\omega_0 > 0$ is given by

$$\omega_0{}^2 = \frac{k_d{}^2 - (p_1^2 + p_2^2)}{2} + \frac{\sqrt{\left(k_d{}^2 - (p_1^2+p_2^2)\right)^2 + 4(k_p^2 - p_1^2 p_2^2)}}{2}. \tag{5}$$

Proposition 2.2. Let $P_\tau(s)$ be given by (3), with $p_1 = p = \alpha + j\beta$, $p_2 = \bar{p} = \alpha - j\beta$, where $\mathrm{Re}(p) = \alpha \geq 0$. Suppose that $k_i = 0$. Then for any given $K_{PID}(s)$ to stabilize $P_0(s)$, it is necessary and sufficient that $k_p > |p|^2$ and $k_d > 2\alpha$. Under these conditions,

$$\bar{\tau}_{PID}(K_{PID}) = \frac{\tan^{-1}\frac{\omega_0 - \beta}{\alpha}}{\omega_0} + \frac{\tan^{-1}\frac{\omega_0+\beta}{\alpha}}{\omega_0} + \frac{\tan^{-1}\frac{k_d\omega_0}{k_p}}{\omega_0} - \frac{\pi}{\omega_0}, \tag{6}$$

where $\omega_0 > 0$ is given by

$$\omega_0^2 = \frac{k_d^2 + 2\beta^2 - 2\alpha^2}{2} + \frac{\sqrt{(k_d^2 + 2\beta^2 - 2\alpha^2)^2 + 4(k_p^2 - |p|^4)}}{2}. \tag{7}$$

In the sequel we shall also need the following lemma, which collects a number of useful properties of the $\tan^{-1}$ function. We refer these properties to Boros and Moll (2005).

Lemma 2.1. Suppose that $\xi \geq 0, \ \eta \geq 0$. Then,

(i) $\tan^{-1}\xi$ is monotonically increasing with ξ.

(ii)

$$\frac{\xi}{1+\xi^2} \leq \tan^{-1}\xi \leq \xi.$$

(iii)

$$\tan^{-1}\xi + \tan^{-1}\eta = \begin{cases} \tan^{-1}\frac{\xi+\eta}{1-\xi\eta}, & \xi\eta < 1, \\ \tan^{-1}\frac{\xi+\eta}{1-\xi\eta} + \pi, & \xi\eta > 1. \end{cases}$$

(iv)

$$\tan^{-1}\xi - \tan^{-1}\eta = \tan^{-1}\frac{\xi - \eta}{1+\xi\eta}.$$

3. MAIN RESULTS

3.1 Real Poles

We first provide the following lower bound on the delay margin $\bar{\tau}_{PID}$ when $P_0(s)$ contains two real unstable poles.

Theorem 3.1. Let $P_\tau(s)$ be given by (3), with $p_1 \geq 0$ and $p_2 \geq 0$ being real poles. Suppose that $k_i = 0$. Then,

$$\bar{\tau}_{PID} \geq \left(\frac{1}{\sqrt{p_1 p_2}}\right) \frac{\tan^{-1} \frac{\theta\sqrt{2+\sqrt{1+2\theta^2}}}{\theta^2+1+\sqrt{1+2\theta^2}}}{\sqrt{2+\sqrt{1+2\theta^2}}} \tag{8}$$

$$\geq \frac{\frac{1}{p_1}+\frac{1}{p_2}}{2+\theta^2+2\sqrt{1+2\theta^2}}\left(1+\frac{1}{\theta^2+\sqrt{1+2\theta^2}}\right), \tag{9}$$

where

$$\theta = \sqrt{\frac{p_1}{p_2}} + \sqrt{\frac{p_2}{p_1}}.$$

Furthermore,

$$\bar{\tau}_{PID} \geq \frac{1}{2}\left(\frac{\frac{1}{p_1}+\frac{1}{p_2}}{5+2\frac{p_1}{p_2}+2\frac{p_2}{p_1}}\right)\left(1+\frac{1}{5+2\frac{p_1}{p_2}+2\frac{p_2}{p_1}}\right). \tag{10}$$

Proof. From Proposition 2.1, we note that

$$\bar{\tau}_{PID} = \sup\left\{\bar{\tau}_{PID}(K_{PID}):\ k_p > p_1 p_2,\ k_d > p_1 + p_2\right\}. \tag{11}$$

In view of Lemma 2.1 (iii), we may write

$$\bar{\tau}_{PID}(K_{PID}) = \frac{\tan^{-1}\frac{k_d}{k_p}\omega_0 - \tan^{-1}\frac{(p_1+p_2)\omega_0}{\omega_0^2-p_1p_2}}{\omega_0}.$$

Setting $k_p = p_1 p_2$ yields $\omega_0^2 = k_d^2 - (p_1^2+p_2^2)$. Define then

$$\overline{g}(k_d) = \frac{\tan^{-1}\frac{k_d\sqrt{k_d^2-(p_1^2+p_2^2)}}{p_1p_2} - \tan^{-1}\frac{(p_1+p_2)\sqrt{k_d^2-(p_1^2+p_2^2)}}{k_d^2-(p_1^2+p_2^2+p_1p_2)}}{\sqrt{k_d^2-(p_1^2+p_2^2)}},$$

$$\underline{g}(k_d) = \frac{\tan^{-1}\frac{(p_1+p_2)\sqrt{k_d^2-(p_1^2+p_2^2)}}{p_1p_2} - \tan^{-1}\frac{(p_1+p_2)\sqrt{k_d^2-(p_1^2+p_2^2)}}{k_d^2-(p_1^2+p_2^2+p_1p_2)}}{\sqrt{k_d^2-(p_1^2+p_2^2)}}.$$

Since $\tan^{-1}(\cdot)$ is monotonically increasing, we have $\overline{g}(k_d) \geq \underline{g}(k_d)$. It thus follows that

$$\begin{aligned}\bar{\tau}_{PID} &\geq \sup\{\overline{g}(k_d):\ k_d > p_1+p_2\} \\ &\geq \sup\{\underline{g}(k_d):\ k_d > p_1+p_2\}.\end{aligned}$$

In what follows we attempt to find the maximum of $\underline{g}(k_d)$ for $k_d > p_1 + p_2$. To facilitate the derivation, we introduce a new variable x such that

$$k_d^2 = p_1^2 + p_2^2 + p_1 p_2 x^2.$$

Evidently, for $k_d > p_1 + p_2$, we have $x > \sqrt{2}$. Define $g(x) = \underline{g}(k_d)\big|_{k_d=\sqrt{p_1^2+p_2^2+p_1p_2x^2}}$. It follows that

$$\sup_{k_d>p_1+p_2} \underline{g}(k_d) = \sup_{x>\sqrt{2}} g(x).$$

By a direct calculation, we obtain

$$g(x) = \frac{\tan^{-1}(\theta x) - \tan^{-1}\frac{\theta x}{x^2-1}}{x\sqrt{p_1p_2}}. \tag{12}$$

Invoking Lemma 2.1 (iv), we may rewrite $g(x)$ as

$$g(x) = \frac{\tan^{-1}\frac{\theta x(x^2-2)}{(\theta^2+1)x^2-1}}{x\sqrt{p_1p_2}}.$$

Furthermore, using Lemma 2.1 (ii), we obtain a lower bound on $g(x)$ as

$$g(x) \geq \left(\frac{\theta}{\sqrt{p_1p_2}}\right)\frac{\left((\theta^2+1)x^2-1\right)(x^2-2)}{\left((\theta^2+1)x^2-1\right)^2+\theta^2x^2(x^2-2)^2},$$

which in turn gives rise to $g(x) \geq \dfrac{\theta}{\sqrt{p_1p_2}}\left(g_1(x)+g_2(x)\right)$, where

$$g_1(x) = \frac{x^2-2}{\theta^2x^2+(x^2-1)^2},$$

$$g_2(x) = \frac{(x^2-2)^2}{(\theta^2x^2+1)(\theta^2x^2+(x^2-1)^2)}.$$

Maximizing $g_1(x)$ yields its maximum at

$$x^* = \sqrt{2+\sqrt{1+2\theta^2}}.$$

The lower bound in (8) is then established by evaluating $g(x^*)$, and the lower bound in (9) is obtained by evaluating $g_1(x^*)+g_2(x^*)$. The bound (9) can be further weakened to

$$\begin{aligned}\bar{\tau}_{PID} &\geq \frac{\theta/\sqrt{p_1p_2}}{2\theta^2+2\sqrt{1+2\theta^2}}\left(1+\frac{1}{\theta^2+\sqrt{1+2\theta^2}}\right) \\ &\geq \frac{1}{2}\left(\frac{\theta/\sqrt{p_1p_2}}{1+2\theta^2}\right)\left(1+\frac{1}{1+2\theta^2}\right),\end{aligned}$$

where the first inequality follows by noting that $\theta^2 \geq 2$, and the second by noting that $\sqrt{1+2\theta^2} \leq 1+\theta^2$. The proof is completed by substituting θ. ■

Remark 3.1. The progressively weakened lower bounds in (8)-(10), from less conservative to more explicit, may serve for different purposes. For a given pair of p_1 and p_2, the bounds in (8) and (9) can be numerically evaluated. The bounds (9) and (10) are better fitted for comparison to earlier results, e.g., those in Ma and Chen (2017); Wei and Lin (2017); Zhou et al. (2012). In this vein, we note the upper bound on $\bar{\tau}_{PID}$ obtained in Ma and Chen (2017):

$$\bar{\tau}_{PID} \leq \frac{\tan^{-1}\left(\sqrt{\frac{2p_2}{p_1}}+\sqrt{\frac{2p_1}{p_2}}\right)}{\sqrt{2p_1p_2}},$$

and that on $\bar{\tau}$ obtained in Ju and Zhang (2016),

$$\bar{\tau} \leq \frac{2\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}{1+\frac{p_1}{p_2}+\frac{p_2}{p_1}},$$

which is contingent on use of general LTI controllers. Note that while in general the bounds (8)-(10) can be conservative, they are nonetheless sharp in the limit, when, for example, one of p_1 and p_2 lies at the origin, i.e., when the plant is given by

$$P_\tau(s) = \frac{1}{s(s-p)}e^{-\tau s}, \qquad p > 0. \tag{13}$$

We summarize this observation in the following corollary.

Corollary 3.2. Let $P_\tau(s)$ be given by (13). Suppose that $k_i = 0$. Then,

$$\bar{\tau}_{PID} \geq \frac{1}{p}. \tag{14}$$

Proof. It follows by setting $p_2 = p$ and taking the limit of the lower bound in (9) with $p_1 \to 0$. ■

Note that the lower bound in (14) was shown to be exact in Ma and Chen (2017).

Also of interest is a plant with a double pole, that is, the plant

$$P_\tau(s) = \frac{1}{(s-p)^2}e^{-\tau s}, \quad p > 0. \tag{15}$$

The following corollary is an immediate consequence of Theorem 3.1, which follows by setting $p_1 = p_2 = p$.

Corollary 3.3. Let $P_\tau(s)$ be given by (15). Suppose that $k_i = 0$. Then,

$$\bar{\tau}_{PID} \geq \left(\frac{\tan^{-1}(\sqrt{5}/4)}{\sqrt{5}}\right)\frac{1}{p} \tag{16}$$

$$\geq \left(\frac{4}{21}\right)\frac{1}{p}. \tag{17}$$

Remark 3.2. In connection with Corollary 3.3, it is useful to recall that for the plant given in (15), the upper bound Ma and Chen (2017)

$$\bar{\tau}_{PID} \leq \left(\frac{\tan^{-1}(2\sqrt{2})}{\sqrt{2}}\right)\frac{1}{p}$$

holds, while more generally, it was found in Ju and Zhang (2016) that

$$\bar{\tau} \leq \left(\frac{4}{3}\right)\frac{1}{p}.$$

3.2 Complex Conjugate Poles

We now develop in parallel the lower bounds for plants that contain a pair of complex conjugate poles. Thus, we consider the system (3) with $p_1 = p = \alpha + j\beta$, $p_2 = \bar{p} = \alpha - j\beta$, where $\mathrm{Re}(p) = \alpha \geq 0$. As stipulated in the preceding section, we consider as well a PD controller given by $K_{PID}(s) = k_p + k_d s$. The following theorem gives an analogous result, whose proof is omitted due to space consideration.

Theorem 3.4. Let $P_\tau(s)$ be given by (3), with $p_1 = p = \alpha + j\beta$, $p_2 = \bar{p} = \alpha - j\beta$, where $\mathrm{Re}(p) = \alpha \geq 0$. Suppose that $k_i = 0$. Then,

$$\bar{\tau}_{PID} \geq \left(\frac{1}{|p|}\right)\frac{\tan^{-1}\frac{\gamma\sqrt{2+\sqrt{1+2\gamma^2}}}{1+\gamma^2+\sqrt{1+2\gamma^2}}}{\sqrt{2+\sqrt{1+2\gamma^2}}} \tag{18}$$

$$\geq \frac{(2\alpha/|p|^2)}{2+\gamma^2+2\sqrt{1+2\gamma^2}}\left(1+\frac{1}{\gamma^2+\sqrt{1+2\gamma^2}}\right), \tag{19}$$

where $\gamma = \dfrac{2\alpha}{|p|}$.

Furthermore,

$$\bar{\tau}_{PID} \geq \left(\frac{4}{21}\right)\frac{\mathrm{Re}(p)}{|p|^2}. \tag{20}$$

Additionally,

$$\bar{\tau}_{PID} \geq \frac{1}{|p|}\left(\sqrt{\sqrt{1+\left(1-2\frac{\alpha^2}{|p|^2}\right)^2}-\left(1-2\frac{\alpha^2}{|p|^2}\right)} - \frac{2(\alpha/|p|)}{\sqrt{1+\left(1-2\frac{\alpha^2}{|p|^2}\right)^2-2\frac{\alpha^2}{|p|^2}}}\right). \tag{21}$$

Remark 3.3. It is worth pointing out that the bounds in (18) and (19) are most useful when $\alpha/|p|$ is relatively large. In the limit when $\alpha = |p|$, i.e., when the conjugate poles p and $\bar{p}$ degenerate to a double real pole, these bounds coincide with those in (16) and (17), respectively. On the other hand, the bound in (21) provides a tighter estimate when $\alpha/|p|$ is small; in the extreme case when $\alpha = 0$, i.e., for a pair of purely imaginary poles, which correspond to the plant

$$P_\tau(s) = \frac{1}{s^2+\omega_c^2}e^{-\tau s}, \qquad \omega_c \geq 0, \tag{22}$$

it leads to the following corollary.

Corollary 3.5. Let $P_\tau(s)$ be given by (22). Suppose that $k_i = 0$. Then,

$$\bar{\tau}_{PID} \geq \left(\sqrt{\sqrt{2}-1}\right)\frac{1}{\omega_c}. \tag{23}$$

It is thus clear that for a double integrator system, i.e., when $\omega_c = 0$, the delay margin achievable by a PID controller can be made infinitely large. This observation corroborates with the results in Wei and Lin (2017); Zhou et al. (2009, 2012), where it was shown that for systems with unstable poles solely at the origin, an arbitrarily large delay margin can be achieved by finite-dimensional state feedback Zhou et al. (2009), and that for systems with non-zero imaginary poles, only finite delay margin can be attained Wei and Lin (2017).

3.3 A Comparison

It is instructive to compare the bounds derived herein to those obtained elsewhere, most of which, however, are upper bounds with pessimism undetermined and consequently are unable to provide a valid comparison. Nevertheless, in the recent work Wei and Lin (2017), an assortment of lower bounds were developed for $\bar{\tau}$, i.e., the delay margin achievable by general LTI controllers. We thus compare our results with those of Wei and Lin (2017).

The design method employed in Wei and Lin (2017) is chiefly based on predictor state feedback, whose bounds generally admit rather sophisticated expressions but can be simplified as follows. In one case the bound can be raised to

$$\frac{1}{2\sqrt{2(n+1)}\sum_{i=1}^{n} p_i},$$

where p_i are the plant unstable poles, and n is the plant order. This case corresponds to scenarios where the delay-free system $P_0(s)$ contains only unstable poles and not all unstable poles p_i lie on the imaginary axis. For systems with solely imaginary poles p_i, an alternative bound can be ramped to

$$\frac{1}{\sqrt{2n\sum_{i=1}^{n}|p_i|^2}},$$

where p_i are purely imaginary poles and n is the plant order. Note that in both cases the bounds in Wei and Lin (2017) may not perform as well, but are elevated for ease of comparison; in other words, the actual bounds in Wei and Lin (2017) cannot attain these quantities but instead, are bounded from above by the two simplified expressions, respectively. The following table provides a comparison of the lower bounds in the corresponding cases. Note that in the case of two real unstable poles, the bound in (9) (and

$P_0(s)$	$\frac{1}{(s-p_1)(s-p_2)}$	$\frac{1}{s(s-p)}$	$\frac{1}{(s-p)^2}$	$\frac{1}{(s-p)(s-\overline{p})}$	$\frac{1}{s^2+\omega_c^2}$
Predictor State Feedback	$\left(\frac{1}{2\sqrt{6}}\right)\frac{1}{p_1+p_2}$	$\left(\frac{1}{2\sqrt{6}}\right)\frac{1}{p}$	$\left(\frac{1}{4\sqrt{6}}\right)\frac{1}{p}$	$\left(\frac{1}{4\sqrt{6}}\right)\left(\frac{1}{(\alpha/\lvert p\rvert)}\right)\frac{1}{\lvert p\rvert}$	$\left(\frac{1}{2\sqrt{2}}\right)\frac{1}{\omega_c}$
PID Output Feedback	(9)	$\frac{1}{p}$	$\left(\frac{4}{21}\right)\frac{1}{p}$	(18) and (21)	$\left(\sqrt{\sqrt{2}-1}\right)\frac{1}{\omega_c}$

hence (8)) is tighter than that in Wei and Lin (2017). To see this, one can easily verify that the function

$$\frac{\theta^2}{2+\theta^2+2\sqrt{1+2\theta^2}}$$

is monotonically increasing, and that for $\theta \geq 2$,

$$\frac{\theta^2}{2+\theta^2+2\sqrt{1+2\theta^2}} \geq \frac{2}{3} > \frac{1}{2\sqrt{6}}.$$

Equivalently, this gives

$$\frac{\frac{1}{p_1}+\frac{1}{p_2}}{2+\theta^2+2\sqrt{1+2\theta^2}} > \left(\frac{1}{2\sqrt{6}}\right)\frac{1}{p_1+p_2}.$$

Hence in all cases except that of complex conjugate poles, the PID controller delivers a larger bound, whereas with respect to complex poles, the limiting cases given in the table ($\alpha = 0$, or $\beta = 0$) show that the bounds obtained by PID control remain advantageous under various circumstances.

4. CONCLUSION

In this paper we have derived explicit lower bounds on the delay margin of second-order unstable delay systems achievable by PID control. Unlike the upper bounds obtained elsewhere, which can be used to determine the range of delay where a delay plant cannot be robustly stabilized, the lower bounds obtained herein serve an opposite purpose: they provide *a priori* the ranges of delay over which the plant is guaranteed to be stabilizable by a PID controller, and hence more generally, by a finite-dimensional LTI controller. Note that explicit, exact expressions of the delay margin have been found previously for first-order systems, while it is generally infeasible, if ever possible, to stabilize plants of an higher-order by PID control, even for plants free of delay.

When confined to second-order systems, our results are seen in most cases to better those obtained elsewhere using general LTI controllers, notably those of Wei and Lin (2017) based on the design of predictor state feedback, which typically results in high-order controllers and are blessed with more degrees of design freedom. The implication is that in the context of using PID control, the earlier bounds obtained elsewhere will be overly conservative compared to the ones derived herein. Notwithstanding this improvement, nonetheless, methods such as the predictor state feedback design are broadly applicable to high-order delay systems containing more unstable poles, while our results are limited to systems with no more than two unstable poles, a limitation that can be attributed to PID control in general.

REFERENCES

Boros, G. and Moll, V.H. (2005). Sums of arctangents and some formulas of Ramanujan. *SCIENTIA Series A: Mathematical Sciences*, 11, 13–24.

Bresch-Pietri, D., Chauvin, J., and Petit, N. (2012). Adaptive control scheme for uncertain time-delay systems. *Automatica*, 48, 1536–1552.

Gaudette, D.L. and Miller, D.E. (2016). Adaptive stabilization of a class of time-varying systems with an uncertain delay. *Mathematics of Control, Signals, and Systems*, 28(3).

Ju, P. and Zhang, H. (2016). Further results on the achievable delay margin using LTI control. *IEEE Transactions on Automatic Control*, 61(10), 3134–3139.

Krstic, M. (2017). On the applicability of PID control to nonlinear second-order systems. *National Science Review*, 4(5), 668. doi:10.1093/nsr/nwx048. URL `http://dx.doi.org/10.1093/nsr/nwx048`.

Liberis, N.B. and Krstic, M. (2013). *Nonlinear Control Under Nonconstant Delays.* SIAM Press, Philadelphia, PA.

Ma, D. and Chen, J. (2017). Delay robustness of low-order systems under pid control. In *Proc. IEEE Conf. Decision and Control*, 4333–4338. Melbourne, Australia.

Michels, W., Engelborghs, K., Vansevenant, P., and Roose, D. (2002). Continuous pole placement for delay equations. *Automatica*, 38, 747–761.

Michiels, W. and Niculescu, S.I. (2007). *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach.* SIAM Press, Philadelphia, PA.

Middleton, R.H. and Miller, D.E. (2007). On the achievable delay margin using lti control for unstable plants. *IEEE Transactions on Automatic Control*, 52, 1194–1207.

Miller, D.E. and Davison, D.E. (2005). Stabilization in the presence of an uncertain arbitrarily large delay. *IEEE Transactions on Automatic Control*, 50(8), 1074–1089.

Qi, T., Zhu, J., and Chen, J. (2017a). Fundamental limits on uncertain delays: When is a delay system stabilizable by LTI controllers?. *IEEE Transactions on Automatic Control*, 62(3), 1314–1328.

Qi, T., Zhu, J., and Chen, J. (2017b). On delay radii and bounds of MIMO systems. *Automatica*, 77, 214–218.

Silva, G.J., Datta, A., and Bhattacharyya, S.P. (2002). New results on the synthesis of PID controllers. *IEEE Transactions on Automatic Control*, 47(2), 241–252.

Silva, G.J., Datta, A., and Bhattacharyya, S.P. (2005). *PID Controllers for Time-Delay Systems.* Birkhuster, Basel.

Wei, Y. and Lin, Z. (2017). Maximum delay bounds of linear systems under delay independent truncated predictor feedback. *Automatica*, 83(9), 65–72.

Yoon, S.Y. and Lin, Z. (2013). Truncated predictor feedback control for exponentially unstable linear systems with time-varying input delay. *Systems and Control Letters*, 62(10), 837–844.

Zhou, B., Lin, Z., and Duan, G.R. (2009). Properties of the parametric lyapunov equation-based low-gain design with applications in stabilization of time-delay systems. *IEEE Transactions on Automatic Control*, 54(7), 1698–

1704.

Zhou, B., Lin, Z., and Duan, G.R. (2012). Truncated predictor feedback for linear systems with long time-varying input delays. *Automatica*, 48(10), 2387–2399.