Exploring the Rendezvous of Agents in Cyclic Pursuit with Possible Negative Controller Gain and Homogeneous Input Delay

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Abstract: In this paper, the effect of homogeneous input delay on rendezvous of a group of agents under cyclic pursuit strategy is investigated. Presence of negative controller gain(s) aids in expanding the reachable set. It is found that at most one negative controller gain is possible for all the agents to converge under delayed condition. In contrast to accommodating any positive controller gains in the no-delay case, the controller gains in delayed case are restricted by the delay. A modification on the gains is proposed in order to tolerate arbitrarily large bounded delay. The weighted centroid of all agents does not change as time evolves and all agents finally converge to their weighted centroid. Simulation results are provided to substantiate the obtained delay-dependent conditions.

Keywords: Cyclic Pursuit, Input Delay, Root Tendency, Single-Integrator.

1. INTRODUCTION

Research in multi-agent systems has drawn considerable attention because of its extensive applications in wireless communication, unmanned aerial vehicle, spacecraft (He et al., 2014; Ren, 2010; Van Der Walle et al., 2008; Mu et al., 2017). The collective behavior in multi-agent system can be found in flocking of birds (Jadbabaie et al., 2003), rendezvous for mobile autonomous robots (Smith et al., 2007) and so on. Consensus algorithm finds its appliance in investigating these collective behaviors. Specifically, rendezvous refers to the motion of a group of agents governed by simple interaction rules with limited environmental information such that all agents reach to a common position simultaneously (Su et al., 2010). In this work, we investigate the conditions for rendezvous in the presence of input delays.

Cyclic pursuit strategy is related to consensus, where each agent follows its immediate neighbour (Ren, 2009; Trinh et al., 2017). The network topology thus forms a directed cyclic graph, and consensus protocols are pertinent. Neighbour-based rules have broad applications in distributed multi-agent system and can be found in Cao et al. (2008); Li et al. (2014). Marshall et al. (2004) analyze the linear cyclic pursuit with homogeneous positive controller gains and establish that centroid of the agents remains stationary and all agents converge to the centroid. However, rendezvous may not occur at the centroid if the gains are heterogeneous (Sinha and Ghose, 2006a,b). Sinha and Ghose (2006a,b) analyze the stability with one negative gain in order to expand the reachable set. Recent work with negative edge weights can be found in Mukherjee and Ghose (2017); Ahmadizadeh et al. (2017). However, the effect of input delay is not considered in these works, and here, we investigate the conditions for rendezvous in the presence of homogeneous input delay.

Input delay is present in multi-agent system due to possible time for actuation of an agent. Presence of input delay may degrade the performance of the multi-agent system and even result in instability (Liu and Liu, 2013; Wang and Ding, 2016). Tian and Liu (2008) discuss the effect of input delays in consensus and achieve some delay-dependent conditions for stability. For a general directed graph, Allen-Prince et al. (2017) find the delay margin which depends on the eigenvalues of the Laplacian matrix. However, present work focuses on deriving conditions on controller gains rather than finding criteria on eigenvalues of Laplacian matrix. Irofti and Atay (2016) consider circulant networks and show that allowable maximum delay for consensus depends on the network topology. All these works consider the edge weights, if existing, to be positive. In the present work, we investigate the delay-dependent conditions with negative controller gain to ensure rendezvous. Sinha and Ghose (2006a,b) establish that at most one negative controller gain is possible without input delay. Yet, the possibility of one or more than one negative controller gains in the presence of input delay has not been examined. This work explores the above issues and we summarize the main contributions as:

- Rendezvous occurs for at most one negative controller gain with other gains strictly positive in the presence of homogeneous input delay. The rendezvous point is independent of the input delay.
- The upper bound on each controller gain is restricted by the input delay. As delay increases, the upper bounds on positive gains decrease which in turn reduces the margin on the negative gain.
- Arbitrarily large but bounded input delay can be accommodated by a proportional change in the controller gains of all agents.
- The weighted centroid of the agents remains stationary and rendezvous occurs at the weighted centroid. Specifically, for homogeneous controller gains the final convergence point is the centroid of agents.

2. LINEAR CYCLIC PURSUIT LAW

We consider the cyclic pursuit problem for a group of n agents. Each agent starts from arbitrary positions in $\mathbb{R}^d$. Agent i receives information from the agent $i+1$ mod n. The communication topology is thereby described by a directed cyclic graph. The kinematics of agent i is

$$\dot{Z}_i(t) = u_i(t-\tau), \tag{1}$$

where $Z_i(t) = \left[z_i^1(t) z_i^2(t) \dots z_i^d(t)\right]^\top \in \mathbb{R}^d$ denotes the position of the agent i and $u_i(t) \in \mathbb{R}^d$ is the control input. The input delay $\tau < \infty$ is constant and uniform for the multi-agent system. The control inputs to achieve rendezvous is based on the relative position of current agent i and its in-neighbour $i+1$ mod n. The control input for agent i has the following form:

$$u_i(t) = k_i \left(Z_{i+1}(t) - Z_i(t)\right). \tag{2}$$

Here, the gains $k_i \in \mathbb{R}$ are finite $(-\infty < k_i < \infty)$. The objective is to achieve a desired rendezvous $\bar{Z}$ in the presence of input delay τ, that is, for arbitrary initial conditions

$$\lim_{t\to\infty} \|\bar{Z} - Z_i(t)\| = 0. \tag{3}$$

We make the following assumption to accomplish the objective:

Assumption 1. Input delay τ is constant, bounded and homogeneous throughout the network.

3. RENDEZVOUS IN CYCLIC PURSUIT

From the control law given by (2), it can be noted that velocity of i^{th} agent is proportional to the relative position of $(i+1)^{\text{th}}$ agent with respect to i^{th} agent. Let us denote m^{th} coordinate of agent i as z_i^m. As the coordinates $m = 1, \dots d$ of each agent evolve independently, the cyclic pursuit problem can be cast into following d identical decoupled linear time-delay systems

$$\begin{bmatrix} \dot{z}_1^m \\ \dot{z}_2^m \\ \vdots \\ \dot{z}_{n-1}^m \\ \dot{z}_n^m \end{bmatrix} = \begin{bmatrix} -k_1 & k_1 & 0 & \cdots & 0 & 0 \\ 0 & -k_2 & k_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -k_{n-1} & k_{n-1} \\ k_n & 0 & 0 & \cdots & 0 & -k_n \end{bmatrix} \begin{bmatrix} z_1^m(t-\tau) \\ z_2^m(t-\tau) \\ \vdots \\ z_{n-1}^m(t-\tau) \\ z_n^m(t-\tau) \end{bmatrix},$$

where $m = 1, 2, \dots d$. As the equation of motion in any coordinate is exactly same, the cyclic pursuit problem of n agents reduces to stability analysis of a single delay differential equation of the form

$$\dot{x}(t) = Ax(t-\tau), \tag{4}$$

$$\text{with} \quad A = \begin{bmatrix} -k_1 & k_1 & 0 & \cdots & 0 & 0 \\ 0 & -k_2 & k_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -k_{n-1} & k_{n-1} \\ k_n & 0 & 0 & \cdots & 0 & -k_n \end{bmatrix}.$$

Here, $(-A)$ represents the Laplacian matrix corresponding to the directed cyclic graph. The characteristic quasipolynomial of (4) is therefore represented as

$$\rho(s) = |sI_n - Ae^{-s\tau}| = \prod_{i=1}^{n} \left(s + k_i e^{-s\tau}\right) - e^{-ns\tau} \prod_{i=1}^{n} k_i. \tag{5}$$

Now, we investigate conditions on the controller gains such that agents meet at a point. For the occurrence of rendezvous, it is necessary that the time-delayed system (4) is stable. The delay-dependent conditions can be obtained with the help of Lemma 1.

Lemma 1. The system (4) with $\tau = 0$ has a simple eigenvalue at the origin and the remaining eigenvalues in the open left-half complex plane if and only if

(i) at most one agent has non-positive controller gain k_γ with other agents having positive controller gains ($k_i > 0$, $i \neq \gamma$), and

(ii) the bound on k_γ is $k_\gamma > -\left(\prod_{\substack{i=1 \\ i\neq\gamma}}^{n} k_i\right) \Bigg/ \left(\sum_{\substack{i=1 \\ i\neq\gamma}}^{n} \prod_{\substack{j=1 \\ j\neq\gamma, i}}^{n} k_j\right)$.

Proof. The proof of the Lemma is discussed in the work of Sinha and Ghose (2006b). ■

Lemma 1 tells us that if more than one controller gain is negative, the agents will never converge to a point in the absence of input delay. However, depending on the stabilizing or destabilizing effect of the delay, rendezvous may or may not be possible with one or more than one negative controller gains . In Lemma 2, we investigate the possibility of negative controller gains for rendezvous.

Lemma 2. Consider a group of agents with kinematics (1)-(2). If more than one controller gain is non-positive, rendezvous is never possible.

Proof. We analyze the characteristics of the roots of (5) when more than one gain is non-positive. This analysis has been done considering two cases.

- **Case 1:** At least two gains are zero.
  Let there be $p(\geq 2)$ zero controller gains and symbolically we denote these gains as $k_{\gamma 1}, k_{\gamma 2}, \dots, k_{\gamma p}$. Under this scenario, $\rho(s)$ can be expressed as $\rho(s) = s^p \prod_{\substack{i=1 \\ i\neq\gamma_1,\dots,\gamma_p}}^{n} \left(s + k_i e^{-s\tau}\right)$. As $p \geq 2$, $\rho(s)$ has at least two roots at origin irrespective of the input delay τ. This implies a polynomial divergence of trajectory with time. Hence, the system (4) is unstable.

- **Case 2:** One gain is negative and at least one gain is non-positive.
At first, we consider the case when there is no input delay. For $\tau = 0$, the characteristic quasipolynomial becomes $\rho(s)|_{\tau=0} = \prod_{i=1}^{n}(s+k_i) - \prod_{i=1}^{n} k_i$. If at least one gain k_γ is zero, then $\rho(s)|_{\tau=0}$ is simplified to $\rho(s)|_{\tau=0} = s\prod_{\substack{i=1\\ i\neq\gamma}}^{n}(s+k_i)$. Hence, for the existence of one negative gain and at least one zero gain, $\rho(s)|_{\tau=0}$ will have at least one root in the open right-half complex plane. Lemma 1 signifies that in the presence of at least two negative controller gains, $\rho(s)|_{\tau=0}$ has either double root at origin or at least one root in the open right-half complex plane. Hence, the system (4) is unstable in the absence of delay for the existence of one negative gain and at least one non-positive gain.
Now, we investigate the stability of the system (4) with one negative gain and at least one non-positive gain in the presence of delay $0 < \tau < \infty$. For stability, the root(s) of $\rho(s)|_{\tau=0}$ with positive real part(s) in the open right complex plane have to cross the imaginary axis from right-half to left-half and $\rho(s)$ must have a simple root at origin. First we consider the case when $\rho(s)|_{\tau=0}$ has at least two roots at the origin. This is possible if and only if

$$\left.\frac{d\left(\rho(s)|_{\tau=0}\right)}{ds}\right|_{s=0} = 0, \quad \Leftrightarrow \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n} k_j = 0. \qquad (6)$$

Satisfying (6), we evaluate $\left.\frac{d\rho(s)}{ds}\right|_{s=0}$ in the presence of delay τ as

$$\left.\frac{d\rho(s)}{ds}\right|_{s=0} = \left[\sum_{i=1}^{n}\left(1-k_i\tau e^{-s\tau}\right)\prod_{\substack{j=1\\ j\neq i}}^{n}\left(s+k_je^{-s\tau}\right) \right.$$
$$\left.\left. + n\tau e^{-ns\tau}\prod_{i=1}^{n}k_i\right]\right|_{s=0}$$
$$= \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}k_j - \tau\sum_{i=1}^{n}k_i\prod_{\substack{j=1\\ j\neq i}}^{n}k_j + n\tau\prod_{i=1}^{n}k_i = \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}k_j = 0.$$

This signifies that if $\left.\frac{d\left(\rho(s)|_{\tau=0}\right)}{ds}\right|_{s=0} = 0$, then $\left.\frac{d\rho(s)}{ds}\right|_{s=0} = 0$ for any $0 < \tau < \infty$. Also, $\rho(s)$ always has a root at origin. Thus, if the system (4) contains at least two eigenvalues at origin for $\tau = 0$, then there will be at least two roots at origin for any delay $0 < \tau < \infty$.
Let us consider the case when $\rho(s)|_{\tau=0}$ has simple root at origin and at least one root in the open right-half complex plane. The characteristic equation of (4) can be written as

$$|sI_n - Ae^{-s\tau}| = 0, \quad \Rightarrow s\prod_{i=2}^{n}\left(s - e^{-\tau s}\lambda_i(A)\right) = 0,$$

where $\lambda_1(A) = 0$. Existence of at least one root of $\rho(s)|_{\tau=0}$ in the open right-half complex plane implies $Re\{\lambda_{i^\star}(A)\} > 0$ for some $i = i^\star$. We define $\lambda_{i^\star}(A) := a + jb$ with $a > 0$ and $b \geq 0$. We have to investigate the possibility of roots of $h(s) = s - (a+jb)e^{-\tau s}$ to cross the imaginary axis from right to left. The root tendency at $s = j\omega$ with ω be the crossing frequency is

$$\begin{aligned}
\mathrm{RT}|_{s=j\omega} &= \mathrm{sgn}\left[Re\left(\left.\frac{ds}{d\tau}\right|_{s=j\omega}\right)\right] \\
&= \mathrm{sgn}\left[Re\left(\left.\frac{-(a+jb)se^{-s\tau}}{1+\tau(a+jb)e^{-s\tau}}\right|_{s=j\omega}\right)\right] \\
&= \mathrm{sgn}\left[Re\left(\left.\frac{-s^2}{1+s\tau}\right|_{s=j\omega}\right)\right] \\
&= \mathrm{sgn}\left[\frac{\omega^2}{1+\omega^2\tau^2}\right] \neq -1.
\end{aligned}$$

There exists no $\omega \in \mathbb{R}$ such that root tendency can be (-1). Therefore, root of $\rho(s)|_{\tau=0}$ having positive real part cannot cross the imaginary axis from right to left. Hence, there will be at least one root of $\rho(s)$ in the open right complex plane for any bounded delay τ if the open right-half root is present for $\tau = 0$. This proves that the system is unstable for one negative gain along with at least one non-positive gain. Under this scenario, agents cannot converge to any rendezvous point. ∎

In Lemma 2, it is established that rendezvous may occur for not more than one negative gain with other gains being positive. However, there exists bound on the negative gain beyond which system (4) may be unstable. Lemma 3 is useful to find the bound on negative gain.

Lemma 3. Consider a group of agents with kinematics (1)-(2), where one agent has negative controller gain k_γ with other agents having positive controller gains ($k_i > 0$, $i \neq \gamma$). The agents will not converge to a point irrespective of delay $0 < \tau < \infty$ if

$$k_\gamma \leq -\left(\prod_{\substack{i=1\\ i\neq\gamma}}^{n}k_i\right)\Bigg/\left(\sum_{\substack{i=1\\ i\neq\gamma}}^{n}\prod_{\substack{j=1\\ j\neq\gamma,i}}^{n}k_j\right). \qquad (7)$$

Proof. We investigate the location of roots of (5) when only one gain k_γ is negative and rest are positive. It is established in Lemma 1 that $\rho(s)|_{\tau=0}$ has either at least one root in the open right-half complex plane or double roots at origin when k_γ satisfies (7). In the proof of Lemma 2, we establish that delay τ has a destabilizing effect and therefore if the system (4) is unstable at $\tau = 0$, it will remain unstable for any $0 < \tau < \infty$. Hence, for the given bound on k_γ, $\rho(s)$ will have either at least one root with positive real part or double roots at origin irrespective of τ. This implies agents will never converge. ∎

Lemma 3 gives necessary condition on the bound on the negative gain. We still have to prove whether agents will converge or not when $k_\gamma > -\left(\prod_{\substack{i=1\\ i\neq\gamma}}^{n}k_i\right)\Big/\left(\sum_{\substack{i=1\\ i\neq\gamma}}^{n}\prod_{\substack{j=1\\ j\neq\gamma,i}}^{n}k_j\right)$

in the presence of input delay. Also, agents with positive controller gains may have some restriction on their gains to ensure stability of (4). We investigate the stability of (4) in the presence of input delay τ with the help of Lemmas 1-3.

Theorem 1. The time-delayed system (4) has a simple eigenvalue at origin and rest eigenvalues in the open left-half complex plane if

(i) at most one agent γ possesses non-positive controller gain k_γ which is bounded as

$$\frac{1}{2\tau} > k_\gamma > -\left(\prod_{\substack{i=1\\i\neq\gamma}}^{n} k_i\right) \Bigg/ \left(\sum_{\substack{i=1\\i\neq\gamma}}^{n}\prod_{\substack{j=1\\j\neq\gamma,i}}^{n} k_j\right), \quad \text{and}$$

(ii) all other agents $i \neq \gamma$ have positive controller gains bounded as

$$\frac{1}{2\tau} > k_i > 0, \quad i \neq \gamma.$$

Proof. The characteristic quasipolynomial of system (4) is given by (5). It is seen from (5) that $\rho(s)$ always has a root at origin. We consider that agent γ has controller gain

$$k_\gamma > -\left(\prod_{\substack{i=1\\i\neq\gamma}}^{n} k_i\right) \Bigg/ \left(\sum_{\substack{i=1\\i\neq\gamma}}^{n}\prod_{\substack{j=1\\j\neq\gamma,i}}^{n} k_j\right) \tag{8}$$

and other agents $i = 1, 2, \dots, n$, $i \neq \gamma$ possess positive gains. According to Lemma 1, the proposed choices of gains ensure that (4) has a simple eigenvalue at origin and other eigenvalues are in the open left-half complex plane for $\tau = 0$. As τ increases there is a possibility that the roots of $\rho(s)$ having negative real parts cross the imaginary axis from left to right resulting in instability. Therefore, we investigate for conditions under which these roots of $\rho(s)$ have negative real parts only.

First, we check the possibility of existence of at least two roots at origin due to increase in delay τ. From the proof of Lemma 2, we get the condition for existence of at least two roots at origin as $\sum_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n} k_j = 0$. However, for the above mentioned bounds on gains, we get

$$k_\gamma > -\left(\prod_{\substack{i=1\\i\neq\gamma}}^{n} k_i\right) \Bigg/ \left(\sum_{\substack{i=1\\i\neq\gamma}}^{n}\prod_{\substack{j=1\\j\neq\gamma,i}}^{n} k_j\right) \quad \Rightarrow \sum_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n} k_j > 0.$$

Therefore, there always exists a simple root of $\rho(s)$ at origin for any value of $\tau \in [0, \infty)$. Hence, the roots can cross the $j\omega$-axis with an increase in delay for $\omega > 0$. Substituting $s = j\omega$, $\omega > 0$ in (5), we get

$$\prod_{i=1}^{n}\left(j\omega + k_i e^{-j\omega\tau}\right) - \prod_{i=1}^{n}\left(k_i e^{-j\omega\tau}\right) = 0$$
$$\Rightarrow \prod_{i=1}^{n}\left(j\omega + k_i e^{-j\omega\tau}\right) = \prod_{i=1}^{n}\left(k_i e^{-j\omega\tau}\right). \tag{9}$$

Taking the absolute magnitude on both sides of (9), we get

$$\prod_{i=1}^{n}\left|j\omega + k_i e^{-j\omega\tau}\right| = \prod_{i=1}^{n}|k_i|$$
$$\Rightarrow \prod_{i=1}^{n}\left|\frac{\sqrt{k_i^2 + \omega^2 - 2\omega k_i \sin\omega\tau}}{k_i}\right| = 1$$
$$\Rightarrow \prod_{i=1}^{n}\left|\sqrt{1 + \frac{\omega^2 - 2\omega k_i \sin\omega\tau}{k_i^2}}\right| = 1. \tag{10}$$

The necessary condition for the roots of $\rho(s)$ to lie on the imaginary axis is given by (10). When $\tau = 0$, the left hand side of (10) is greater than 1 as $\omega > 0$. Now, we investigate the delay-dependent condition such that roots of $\rho(s)$ never cross the imaginary axis with increase in τ. This is possible if (10) is never satisfied with increase in delay. In other words, given a τ, all roots of $\rho(s)$ will be in the open left-half complex plane with a simple eigenvalue at origin if we design the control gains in a way such that the left hand side of (10) remains greater than 1, $\forall\omega > 0$.

Thus, the stability problem reduces to designing k_i such that

$$\prod_{i=1}^{n}\left|\sqrt{1 + \frac{\omega^2 - 2\omega k_i \sin\omega\tau}{k_i^2}}\right| > 1, \quad \forall\omega > 0. \tag{11}$$

The inequality (11) is satisfied if

$$\left|\sqrt{1 + \frac{\omega^2 - 2\omega k_i \sin\omega\tau}{k_i^2}}\right| > 1, \quad \forall\omega > 0, \quad \forall i,$$
$$\Rightarrow \omega - 2k_i \sin\omega\tau > 0, \quad \forall\omega > 0, \quad \forall i. \tag{12}$$

For gains $k_i (> 0)$, $i \neq \gamma$, the inequality (12) is always satisfied if $\sin\omega\tau$ is nonpositive. For positive value of $\sin\omega\tau$, we get the constraint on k_i as $k_i < \dfrac{1}{2\tau\frac{\sin\omega\tau}{\omega\tau}}$, $\forall\omega > 0$. However, $\max_{\omega\tau\in\mathbb{R}^+} \sin\omega\tau$ is 1. Therefore, if we design the gains k_i as

$$0 < k_i < \frac{1}{2\tau}, \quad \forall i, i \neq \gamma, \tag{13}$$

(12) will be satisfied $\forall i$, $i \neq \gamma$.

Now we focus on deriving condition on the gain k_γ for stability. We already establish that if $0 < k_\gamma < \frac{1}{2\tau}$, (12) is satisfied. When k_γ takes negative value and $\sin\omega\tau$ is nonnegative, $(\omega - 2k_\gamma \sin\omega\tau)$ is always greater than zero. When $\sin\omega\tau$ is negative, condition imposed on k_γ is

$$k_\gamma > \frac{1}{2\tau\frac{\sin\omega\tau}{\omega\tau}}, \quad \forall\omega > 0. \tag{14}$$

From (8) and (14), we get the lower bound on k_γ as

$$k_\gamma > \max\left\{ -\frac{\prod_{\substack{i=1\\i\neq\gamma}}^{n} k_i}{\sum_{\substack{i=1\\i\neq\gamma}}^{n}\prod_{\substack{j=1\\j\neq\gamma,i}}^{n} k_j}, \frac{1}{2\tau \min\limits_{\substack{\omega>0\\ \sin\omega\tau<0}} \dfrac{\sin\omega\tau}{\omega\tau}} \right\}$$

$$= -\min\left\{ \frac{1}{\sum\limits_{\substack{i=1\\ i\neq\gamma}}^{n} \frac{1}{k_i}}, \frac{1}{2\tau \left| \max\limits_{\substack{\omega>0\\ \sin\omega\tau<0}} \frac{\sin\omega\tau}{\omega\tau} \right|} \right\}. \qquad (15)$$

The gains k_i, $i \neq \gamma$ are bounded by delay τ. From (13), we get

$$\sum_{\substack{i=1\\ i\neq\gamma}}^{n} \frac{1}{k_i} > 2(n-1)\tau > 2\tau \left| \max_{\substack{\omega>0\\ \sin\omega\tau<0}} \frac{\sin\omega\tau}{\omega\tau} \right| \qquad (16)$$

Using (15) and (16), we achieve

$$k_\gamma > -\frac{1}{\sum\limits_{\substack{i=1\\ i\neq\gamma}}^{n} \frac{1}{k_i}} = -\left(\prod_{\substack{i=1\\ i\neq\gamma}}^{n} k_i \right) \Bigg/ \left(\sum_{\substack{i=1\\ i\neq\gamma}}^{n} \prod_{\substack{j=1\\ j\neq\gamma,i}}^{n} k_j \right). \qquad (17)$$

The inequality (17) signifies that lower bound on k_γ is decided by the condition imposed on k_γ for no-delay case. Hence, $\frac{1}{2\tau} > k_\gamma > -\left(\prod_{\substack{i=1\\ i\neq\gamma}}^{n} k_i \right) \Big/ \left(\sum_{\substack{i=1\\ i\neq\gamma}}^{n} \prod_{\substack{j=1\\ j\neq\gamma,i}}^{n} k_j \right)$ with other gains $0 < k_i < \frac{1}{2\tau}$ satisfy (12). Therefore, $\rho(s)$ has a simple root at origin and other root are in the open left-half complex plane. ■

Remark 1. From (16) and (17), we get

$$k_\gamma > -k_\gamma^\star = -\left(\sum_{\substack{i=1\\ i\neq\gamma}}^{n} \frac{1}{k_i} \right)^{-1} > -\frac{1}{2(n-1)\tau}.$$

As τ increases, $k_\gamma^\star$ decreases. This is due to the decrease in upper bound of positive controller gains and the very relationship among the margin of k_γ with other gains.

Theorem 2. Consider n agents with kinematics (1)-(2). The weighted centroid of the positions of n agents remains stationary for all time $t \geq 0$. For the controller gains as per Theorem 1, rendezvous occurs at the weighted centroid given by

$$\bar{Z} = \left(\sum_{i=1}^{n} \frac{1}{k_i} Z_i(0) \right) \Bigg/ \left(\sum_{i=1}^{n} \frac{1}{k_i} \right). \qquad (18)$$

Proof. According to Theorem 1, the proposed conditions on controller gains ensure all the eigenvalues of the characteristic equation of system (4) are in the open left-half complex plane with a simple eigenvalue at origin. Therefore, the steady state solution of (4) is governed by the eigenvector corresponding to the simple eigenvalue at the origin. As the eigenvector corresponding to the eigenvalue at origin is $e = [\underbrace{1\,1\,\ldots\,1}_{n-\text{times}}]^\top$, x will converge to ϵe, $\epsilon \in \mathbb{R}$. As the motion of agents are decoupled in each coordinate, all agents finally converge to a point.

Let us find the analytical expression of the rendezvous point $\bar{Z}$. From (1) and (2), we get

$$\frac{\dot{Z}_i(t)}{k_i} = Z_{i+1}(t-\tau) - Z_i(t-\tau), \quad \forall i, \quad \forall t \geq 0$$

$$\sum_{i=1}^{n} \frac{\dot{Z}_i(t)}{k_i} = 0,\ \forall t \geq 0 \Rightarrow \sum_{i=1}^{n} \frac{Z_i(t)}{k_i} = c,\ \forall t \geq 0, \qquad (19)$$

c being a constant. The equality (19) signifies that the weighted centroid $\left(\sum_{i=1}^{n} \frac{Z_i(t)}{k_i} \right) \Big/ \left(\sum_{i=1}^{n} \frac{1}{k_i} \right)$ is stationary $\forall t \geq 0$. As all agents finally converge to $\bar{Z}$, c becomes equal to $\bar{Z} \sum_{i=1}^{n} \frac{1}{k_i}$. Therefore, the rendezvous point is given by

$$\bar{Z} \sum_{i=1}^{n} \frac{1}{k_i} = \sum_{i=1}^{n} \frac{Z_i(t)}{k_i} = c, \quad \forall t \geq 0$$

$$\Rightarrow \bar{Z} = \left(\sum_{i=1}^{n} \frac{1}{k_i} Z_i(0) \right) \Bigg/ \left(\sum_{i=1}^{n} \frac{1}{k_i} \right). \qquad ■$$

Remark 2. From (18), it can be seen that the proportional change in the controller gains does not affect the rendezvous point. Theorem 2 establishes that rendezvous point does not depend on the input delay. However, due to the destabilizing effect of delay the controller gains designed for no-delay case may lead to instability. According to Theorem 1, each gain has an upper bound. Therefore, proportional change in the controller gains can be done that guarantees stability and the modified gains ensure that all agents will meet at the desired rendezvous point.

Remark 3. For homogeneous controller gains ($k_i = k$, $\forall i$) with $0 < k < \frac{1}{2\tau}$, the centroid of the agents remains stationary $\forall t \geq 0$ and the final convergence point is

$$\bar{Z} = \left(\sum_{i=1}^{n} \frac{1}{k} Z_i(0) \right) \Bigg/ \left(\frac{n}{k} \right) = \left(\sum_{i=1}^{n} Z_i(0) \right) \Bigg/ n.$$

4. SIMULATION RESULTS

In this section, we verify the obtained delay-dependent conditions for a group of 4 agents. They move in $x-y$ plane. Initial position of agents $1, \ldots, 4$ are $(-9, 9)$, $(9, 9)$, $(13, -1)$, $(-9, -9)$, respectively. The homogeneous input delay is $\tau = 1$. According to Theorem 1, we design the gains as: $k_1 = 0.4$, $k_2 = 0.1$, $k_3 = 0.1$ and $k_4 = 0.3$.

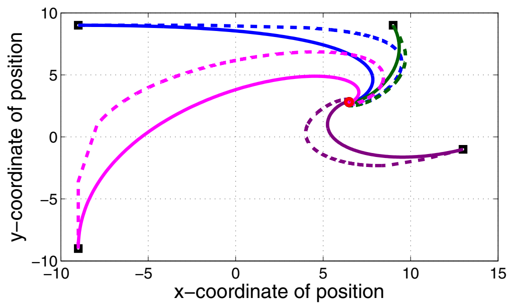

Fig. 1. Trajectory of the agents for positive controller gains $k_1 = 0.4$, $k_2 = 0.1$, $k_3 = 0.1$, $k_4 = 0.3$. Solid lines and dashed lines denote the trajectories of agents without delay and with $\tau = 1$ respectively.

To verify the invariance of the rendezvous point, we study the multi-agent system for both the no-delay and delayed cases. The trajectories of four agents are shown in Fig. 1. It can be seen that the final convergence point $(6.4839, 2.8065)$ is same in both the cases (in accordance with Theorem 2). This signifies that the rendezvous point is invariant with respect to input delay. However, due to the input delay the trajectories are not identical.

Next, we study the system behavior with negative controller gain of agent 2 and other gains same as before. We set $k_2 = -0.05$ satisfying Theorem 1. The trajectories for delayed case and no-delay case are shown in Fig. 2. All agents converge at $(24.6, 47.4)$. The rendezvous point does not change with delay as earlier. Presence of delay restricts the controller gains for a stable rendezvous.

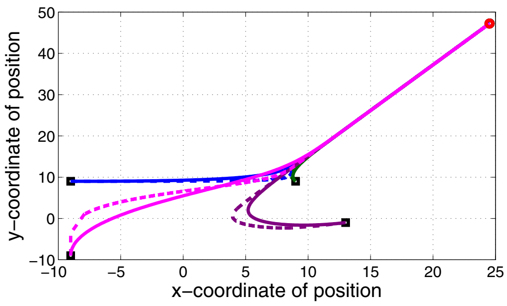

Fig. 2. Trajectory of the agents for controller gains $k_1 = 0.4$, $k_2 = -0.05$, $k_3 = 0.1$, $k_4 = 0.3$. Solid lines and dashed lines denote the trajectories of agents without delay and with $\tau = 1$ respectively.

5. CONCLUSION

In this work, possibility of negative controller gain(s) in cyclic pursuit strategy is investigated and we found that at most one negative controller gain is permissible in the presence of homogeneous input delay. The bounds on the controller gains depend on the input delay. The gains can be appropriately proportioned to accommodate arbitrarily large bounded delay. The weighted centroid remains stationary for all time and rendezvous occurs at the weighted centroid.

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