

# On the stabilization of a system of neutral type occurring in co-generation

Vladimir Răsvan, Daniela Danciu, Dan Popescu

Department of Automation and Electronics, University of Craiova, A.I.Cuza,  
13, Craiova, RO-200585, Romania  
(e-mail: {vrasvan, ddanciu, dpopescu}@automation.ucv.ro)

**Abstract:** There is considered a system of conservation laws with non-standard boundary conditions. At a certain level of linearization, a bilinear controlled system of functional equations is associated by integrating the Riemann invariants of the system along its characteristics. For this associated system the basic theory (existence, uniqueness and smooth data dependence) is developed. Then some invariant set accounting for the positiveness of the variables with physical significance is obtained. Further, its equilibria are shown to be stable but not asymptotically stable as suggests the Stability Postulate of N. G. Četaev. Feedback asymptotic stabilization is obtained by using a suitably designed Lyapunov functional. Using the representation formulae for the solutions, all properties and results thus obtained are projected back on the boundary value problem with bilinear control at the boundaries. This shows a way to obtain stability by the first approximation for the linearized conservation laws with nonlinear boundary conditions.

**Keywords:** Distributed parameter systems, Nonlinear systems, Time delay, Conservation laws, Lyapunov functional, Stabilization

## 1. PROBLEM STATEMENT AND ITS STARTING MATHEMATICAL MODEL

Co-generation used to be (and still is) a valuable source of necessary energy for industrial and individual consumption ensuring an improved efficiency for the energy suppliers. From the technological point of view large scale co-generation is ensured by steam turbines with regulated steam extractions.

We shall focus on the standard case of a low/medium power steam turbine without re-heating and having a single regulated steam extraction. From the control point of view this is a two-variable control object i.e. it has two controlled technological parameters - the rotating speed and the steam pressure at the steam extraction.

The steam flow in the turbine is critical and depends on the product of the controlled cross-section and the upstream steam pressure. This bi-linearity is avoided for the steam input of the turbine by assuming a strictly regulated upstream pressure; for this reason the models of the steam turbines without regulated extractions are assumed linear - see e.g. the IEEE prescriptions Report (1973). However the controlled steam flow at the input of the low pressure cylinder has to ensure namely the steam pressure control and, therefore, it introduces the bilinear term. These considerations led to the following bilinear controlled system

$$\begin{aligned} T_a \frac{ds}{dt} &= \alpha \pi_1 + (1 - \alpha) \pi_2 - v_g, \quad T_1 \frac{d\pi_1}{dt} = \mu_1(t) - \pi_1 \\ T_s \frac{d\pi_s}{dt} &= \pi_1 - \mu_2(t) \pi_s - \beta_1 \pi_s, \quad 0 \leq \mu_1 \leq 1 \\ T_2 \frac{d\pi_2}{dt} &= \mu_2(t) \pi_s - \pi_2, \quad 0 < \mu_{2min} \leq \mu_2 \leq 1 \end{aligned} \quad (1)$$

The model, written in rated (p.u. - *per unit*) variables, has been proposed and used in some previously published references Halanay and Răsvan (1980). The significance of the notations is as follows:  $s$  - rated speed deviation with respect to the synchronous reference speed;  $\pi_1, \pi_s, \pi_2$  - rated steam pressures in the high pressure cylinder, steam extraction chamber and low pressure chamber respectively;  $v_g$  - rated mechanical load at turbine's shaft;  $\mu_k(t)$  - rated cross-sections of the steam flow at the high pressure and low pressure cylinders inputs respectively (control input signals).

Model (1) was thus established as a standard one. A rather different case occurs when the thermal energy consumer is located rather distantly with respect to the power plant and the flow propagation phenomena cannot be neglected any longer. Consequently the steam pressure is controlled at one boundary but the perturbation (consumption) is at the other one. This case has been considered in the pioneering paper of Kabakov (1946), being reproduced in Popov (1954). The aforementioned references have to be mentioned for their modeling aspects: physical laws and choice of the rated variables. Starting from the equations of the isentropic flow e.g. Courant and Hilbert (1966) there are obtained the following partial differential equations for the rated steam pressure, steam flow and for the rated pipe length - see Răsvan (1981)

$$\psi_c T_c \partial_t \xi_\rho + \partial_\lambda \xi_w = 0, \quad \psi_c T_c \partial_t \xi_w + \partial_\lambda \left( \xi_\rho + \frac{\xi_w^2}{\psi_c^2 \xi_\rho} \right) = 0 \quad (2)$$

where  $\xi_\rho(\lambda, t)$  and  $\xi_w(\lambda, t)$  are the rated steam pressure and steam flow respectively. The pipe length is rated hence  $0 \leq \lambda \leq 1$ .

Clearly (2) describes a system of conservation laws. The boundary conditions for (2) account for the coupling to the steam

extraction (at  $\lambda = 0$ ) and to the steam consumer (at  $\lambda = 1$ ). Usually the steam flow entering heat exchangers is subcritical but in our case, where the steam consumer is located rather distantly, the flow might be critical. Description of the subcritical/critical flow has the form - in our case at  $\lambda = 0$  (with the aforementioned notations)

$$\xi_w(0,t) = \pi_s(t)\Phi(\pi_s(t)/\xi_\rho(0,t)) \quad (3)$$

and of the critical flow at  $\lambda = 1$

$$\xi_w(1,t) = \psi_s \xi_\rho(1,t) \quad (4)$$

For the function  $\Phi(x)$  in (3) one can use the Saint Venant formula in the isothermal case

$$\Phi(x) = \begin{cases} (1/x)\sqrt{2\ln x}, & 1 \leq x \leq \sqrt{e} \\ 1/\sqrt{e}, & x \geq \sqrt{e} \end{cases} \quad (5)$$

or the ASME formula as follows

$$\Phi(x) = \begin{cases} x\sqrt{1-x}, & 0 \leq x \leq 2/3 \\ 2/(3\sqrt{3}), & 2/3 \leq x \leq 1 \end{cases} \quad (6)$$

(in this case  $x = \xi_\rho(0,t)/\pi_s(t)$ )

The set of equations (1) - (4) with  $\Phi$  defined either by (5) or (6) defines a nonlinear boundary value problem (with non-standard boundary conditions) for a system of two nonlinear conservation laws. This boundary value problem has some specific features. First, it is non-standard since the boundary condition (3) is controlled - *via* the variable  $\pi_s$  - by the bilinear system of ordinary differential equations (1), itself controlled by the boundary condition (3), thus displaying some kind of internal feedback and this feedback might be a source of instability Neymark (1978). Next, the equations (1) contain the control signals  $\mu_k(t)$ ,  $k = 1, 2$ ; consequently the boundary value problem (1) - (4) is modeling a system with distributed parameters and boundary control. And finally the nonlinear system of conservation laws may induce all possible complex behavior connected to such equations Bressan (2000); Dafermos (2010); Lax (1987, 2006); Serre (2000): spontaneous appearance of the propagating singularities leading to shock formation, rarefaction waves etc.

In power generation systems such phenomena have to be avoided: operating points i.e. equilibria are designed far away of these unpleasant regimes and their stability should be ensured by feedback control.

The considered application described by (1) - (4) will be tackled having in mind the aforementioned aspects. Reasonable linearization will constitute a basis for the approach: the equations are not linearized around some solution but the nonlinear term in (2) i.e.  $\partial_\lambda(\xi_w^2/(\psi_c^2 \xi_\rho))$  is neglected, being negligible in real cases as documented by practical engineers e.g. Kabakov (1946). Consequently the conservation become now a linear system of partial differential equations but with nonlinear boundary conditions.

Starting from this point what is left of the paper is structured as follows. Based on our concept of *augmented validation* Răsvan (2014) that integrates well posedness in the sense of J. Hadamard (existence, uniqueness, and smooth data dependence) with possible existence of invariant sets and inherent stability of the equilibria - a consequence of the *Stability Postulate of N.G. Četaev* - it is considered first a general result of existence and uniqueness, to start with. Further, integration of the Riemann invariants of the linearized (2) along the characteristics will allow the association of a nonlinear coupled de-

lay differential and difference system. A one-to-one correspondence between the solutions of the two mathematical objects is established Răsvan (2014) as resulting from the ideas of A.D. Myshkis - see e.g. Abolinia and Myshkis (1960) - and from those of K.L. Cooke - see Cooke (1970); Cooke and Krumme (1968). Since all results obtained for one object are projected back on the other, we shall focus on the system of functional differential equations for more detailed results on the line of the augmented validation. Its difference part is subsequently linearized around some equilibrium to obtain existence, uniqueness and data dependence, all *via* solution construction by steps. For this bilinear system of functional differential and difference equations (which turns to be of neutral type) an invariant set is displayed, accounting for positiveness of some state variables representing steam pressures. This positiveness will turn useful in control synthesis. The aforementioned synthesis is done using a suitably "guessed" Lyapunov functional. After being synthesized, the control functions are substituted in the initial nonlinear system as well as in the linearized one. For these closed loop systems we dispose of the same Lyapunov functional; the bilinear closed loop system (the "linearized" one) is globally asymptotically stable. Stability by the first approximation of the nonlinear one could be obtained provided a Persidskii type result (uniform asymptotic stability implies exponential stability) could be established. All this subsequent analysis will be continued elsewhere.

## 2. THE SYSTEM OF FUNCTIONAL EQUATIONS

A. We re-write here the equations (1) - (4) with the linearized conservation laws

$$\psi_c T_c \partial_t \xi_\rho + \partial_\lambda \xi_w = 0, \quad \psi_c T_c \partial_t \xi_w + \partial_\lambda \xi_\rho = 0$$

$$\xi_w(0,t) = \pi_s(t)\Phi(\pi_s(t)/\xi_\rho(0,t)), \quad \xi_w(1,t) = \psi_s \xi_\rho(1,t)$$

$$T_a \frac{ds}{dt} = \alpha \pi_1 + (1 - \alpha) \pi_2 - v_g, \quad T_1 \frac{d\pi_1}{dt} = \mu_1(t) - \pi_1$$

$$T_s \frac{d\pi_s}{dt} = \pi_1 - \mu_2(t) \pi_s - \beta_1 \xi_w(0,t), \quad T_2 \frac{d\pi_2}{dt} = \mu_2(t) \pi_s - \pi_2 \quad (7)$$

The equation of  $\pi_s$  has been modified by replacing  $\beta_1 \pi_s$  by  $\beta_1 \xi_w(0,t)$ . The term represents (in rated variables) the steam flow going to the steam consumer and its expression depends on the character of the flow: for critical flows it depends on the upstream pressure  $\pi_s$  only; this had been assumed in standard models to obtain linear ones. During transients and/or in various steady states the flow is subcritical and it will depend on both upstream and downstream local pressure. Therefore the local flow  $\xi_w(0,t)$  at  $\lambda = 0$  was introduced and the boundary conditions at  $\lambda = 0$  take into account both cases. The term  $\mu_2(t) \pi_s$  stands for the controlled flow that "goes" to the Low Pressure turbine cylinder and is always considered to be critical hence it will stay unchanged.

For this system the following results are available: well posedness in the class of both classical and generalized solutions - see the results of Moroşanu (1988). Another property for (7) is existence of an invariant set. Since  $\pi_1$ ,  $\pi_2$ ,  $\pi_s$  represent pressures, they should be positive. The following result, partially known from previous papers can be proved

*Proposition 1.* Consider system (7) with positive coefficients  $0 < \alpha < 1$ ,  $\beta_1 > 0$  and positive control signals - see (1). If  $\pi_1(0) \geq 0$ ,  $\pi_s(0) \geq 0$ ,  $\pi_2(0) \geq 0$  then either  $\pi_1(t) \equiv 0$ ,  $\pi_s(t) \equiv 0$ ,

$\pi_2(t) \equiv 0$  or  $\pi_1(t) > 0$ ,  $\pi_s(t) > 0$ ,  $\pi_2(t) > 0$  for the entire set of the existence of the solution.

*Outline of proof* The proof is direct and makes use of the variation of constants formula as well as a theorem in Bellman (1960), Chapter 10, § 15. We give some detail in what concerns  $\pi_s$ : substituting  $\xi_w(0, t)$  from the boundary conditions we find

$$T_s \frac{d\pi_s}{dt} = \pi_1 - (\mu_2(t) + \beta_1 \Phi(\pi_s(t)/\xi_\rho(0, t))) \pi_s(t) = -a(t)\pi_s + \pi_1(t) \quad (8)$$

where  $\pi_1(t) > 0$  or  $\pi_1(t) \equiv 0$  and  $a(t)$  is bounded. Using the variation of constants formula, the property for  $\pi_s(t)$  follows.

**B.** In order to linearize the boundary conditions around some steady state, we compute a steady state for (7). The steady state values  $\bar{\xi}_\rho$  and  $\bar{\xi}_w$  are constant with respect to  $\lambda$ . We have further  $\bar{\pi}_1 = \bar{\mu}_1$ ,  $\bar{\pi}_2 = \bar{\mu}_2 \bar{\pi}_s^0$  and

$$\alpha \bar{\mu}_1 + (1 - \alpha) \bar{\mu}_2 \bar{\pi}_s^0 = v_g, \quad \bar{\mu}_1 - \bar{\mu}_2 \bar{\pi}_s^0 - \beta_1 \bar{\xi}_w = 0, \quad (9)$$

$$\bar{\xi}_w = \bar{\pi}_s^0 \Phi(\bar{\pi}_s^0 / \bar{\xi}_\rho) = \psi_s \bar{\xi}_\rho$$

Since  $v_g$  - the mechanical load - and  $\bar{\pi}_s^0$  - the required regulated steam pressure - are given, together with  $0 < \psi_s < 1$  which accounts for the thermal load, we have to compute  $\bar{\xi}_\rho$ , then  $\bar{\xi}_w$  and, finally, the reference control values  $\bar{\mu}_1$  and  $\bar{\mu}_2$ . Consider first  $\Phi$  given by (5): if  $\Phi = 1/\sqrt{e}$  then  $\bar{\pi}_s^0 / \bar{\xi}_\rho = \psi_s \sqrt{e}$  and this ratio is higher than  $\sqrt{e}$  only if  $\psi_s > 1$ . Since  $\psi_s < 1$  the equilibrium is located on the other branch of  $\Phi$  given by (5). Therefore

$$\bar{\xi}_\rho = \frac{1}{\sqrt{e\psi_s^2}} \bar{\pi}_s^0, \quad \bar{\xi}_w = \frac{\psi_s}{\sqrt{e\psi_s^2}} \bar{\pi}_s^0 \quad (10)$$

$$\bar{\mu}_1 = v_g + \frac{(1 - \alpha)\beta_1 \psi_s}{\sqrt{e\psi_s^2}} \bar{\pi}_s^0, \quad \bar{\mu}_2 = \frac{v_g}{\bar{\pi}_s^0} - \frac{\alpha\beta_1 \psi_s}{\sqrt{e\psi_s^2}} \bar{\pi}_s^0$$

If the restrictions on  $\bar{\mu}_1$ ,  $\bar{\mu}_2$  are taken into account, then

$$0 \leq v_g + \frac{(1 - \alpha)\beta_1 \psi_s}{\sqrt{e\psi_s^2}} \bar{\pi}_s^0 \leq 1, \quad \mu_{2min} \leq \frac{v_g}{\bar{\pi}_s^0} - \frac{\alpha\beta_1 \psi_s}{\sqrt{e\psi_s^2}} \bar{\pi}_s^0 \leq 1 \quad (11)$$

represent the so called *consumption diagrams*, well known to the turbine design engineers who take care to ensure them since the design stage.

Consider now  $\Phi$  given by (6). If  $\Phi = 2/(3\sqrt{3})$  then  $\bar{\pi}_s^0 / \bar{\xi}_\rho = \psi_s(3\sqrt{3})/2$ . This is possible provided  $\psi_s \sqrt{3} < 1$  i.e. for small enough  $\psi_s$ . If this is the case, then

$$\bar{\xi}_\rho = \frac{2}{3\sqrt{3}} \bar{\pi}_s^0, \quad \bar{\xi}_w = \frac{2\psi_s}{3\sqrt{3}} \bar{\pi}_s^0 \quad (12)$$

$$\bar{\mu}_1 = v_g + \frac{2(1 - \alpha)\beta_1 \psi_s}{3\sqrt{3}} \bar{\pi}_s^0, \quad \bar{\mu}_2 = \frac{v_g}{\bar{\pi}_s^0} - \frac{2\alpha\beta_1 \psi_s}{3\sqrt{3}} \bar{\pi}_s^0$$

On the other branch of this function  $\Phi$  it will follow that  $\bar{\xi}_\rho / \bar{\pi}_s^0 = 1 - \psi_s^2$ ; this is possible provided  $\psi_s \sqrt{3} > 1$  i.e. for  $\psi_s \in (1/\sqrt{3}, 1)$ . Therefore, if this is the case

$$\bar{\xi}_\rho = (1 - \psi_s^2) \bar{\pi}_s^0, \quad \bar{\xi}_w = \psi_s (1 - \psi_s^2) \bar{\pi}_s^0$$

$$\bar{\mu}_1 = v_g + (1 - \alpha)\beta_1 \psi_s (1 - \psi_s^2) \bar{\pi}_s^0, \quad (13)$$

$$\bar{\mu}_2 = \frac{v_g}{\bar{\pi}_s^0} - \alpha\beta_1 \psi_s (1 - \psi_s^2) \bar{\pi}_s^0$$

The next step would be to write (7) in deviations with respect to the aforementioned steady state. The deviation variables are

as follows

$$x_k = \pi_k - \bar{\pi}_k, \quad u_k = \mu_k - \bar{\mu}_k, \quad k = 1, 2; \quad x_s = \pi_s - \bar{\pi}_s^0; \quad (14)$$

$$\zeta_w(\lambda, t) = \xi_w(\lambda, t) - \bar{\xi}_w, \quad \zeta_\rho(\lambda, t) = \xi_\rho(\lambda, t) - \bar{\xi}_\rho$$

and  $s$  is already a rated deviation of the turbine rotating speed.

As an illustrating example, we shall linearize the boundary condition at  $\lambda = 0$  for  $\Phi$  given by (5). We shall have therefore the following bilinear system in deviations

$$\psi_c T_c \partial_t \zeta_\rho + \partial_\lambda \zeta_w = 0, \quad \psi_c T_c \partial_t \zeta_w + \partial_\lambda \zeta_\rho = 0$$

$$\psi_s \zeta_w(0, t) + (1 - \psi_s^2) \zeta_\rho(0, t) = \sqrt{e^{-\psi_s^2}} x_s$$

$$\zeta_w(1, t) = \psi_s \zeta_\rho(1, t)$$

$$T_a \frac{ds}{dt} = \alpha x_1 + (1 - \alpha) x_2, \quad T_1 \frac{dx_1}{dt} = -x_1 + u_1(t) \quad (15)$$

$$T_s \frac{dx_s}{dt} = x_1 - \bar{\mu}_2 x_s - \beta_1 \zeta_w(0, t) - (\pi_s^0 + x_s) u_2(t)$$

$$T_2 \frac{dx_2}{dt} = \bar{\mu}_2 x_s - x_2 + (\pi_s^0 + x_s) u_2(t)$$

**C.** We introduce now the Riemann invariants

$$\zeta^\pm(\lambda, t) = \zeta_\rho(\lambda, t) \pm \zeta_w(\lambda, t) \quad (16)$$

which are subject to the equations

$$\psi_c T_c \partial_t \zeta^\pm \pm \partial_\lambda \zeta^\pm = 0 \quad (17)$$

We integrate  $\zeta^+(\lambda, t)$  along the increasing characteristic  $t^+(\sigma; \lambda, t) = t + \psi_c T_c(\sigma - \lambda)$  to obtain

$$\zeta^+(\lambda, t) = \zeta^+(1, t + \psi_c T_c(1 - \lambda)) \quad (18)$$

hence  $\zeta^+(0, t) = \zeta^+(1, t + \psi_c T_c)$  and  $\zeta^-(\lambda, t)$  along the decreasing characteristic  $t^-(\sigma; \lambda, t) = t - \psi_c T_c(\sigma - \lambda)$  to obtain

$$\zeta^-(\lambda, t) = \zeta^-(0, t + \psi_c T_c \lambda) \quad (19)$$

hence  $\zeta^-(1, t) = \zeta^-(0, t + \psi_c T_c)$  Denoting

$$u^+(t) := \zeta^+(1, t), \quad u^-(t) := \zeta^-(0, t) \quad (20)$$

the linearized boundary conditions will give the following difference system

$$(1 + \psi_s - \psi_s^2) u^+(t + \psi_c T_c) + (1 - \psi_s - \psi_s^2) u^-(t) = 2\sqrt{e^{-\psi_s^2}} x_s(t)$$

$$(1 + \psi_s) u^-(t + \psi_c T_c) = (1 - \psi_s) u^+(t)$$

We denote further

$$\eta^\pm(t) := u^\pm(t + \psi_c T_c); \quad \rho_1 := -\frac{1 - \psi_s - \psi_s^2}{1 + \psi_s - \psi_s^2}, \quad \rho_2 = \frac{1 - \psi_s}{1 + \psi_s};$$

$$\beta_2 = \frac{(1 - \rho_1)(1 + \rho_2)^2}{4\rho_2} \sqrt{e^{-\psi_s^2}}$$

to obtain finally the system of coupled delay differential and difference equations

$$T_a \frac{ds}{dt} = \alpha x_1 + (1 - \alpha) x_2, \quad T_1 \frac{dx_1}{dt} = -x_1 + u_1(t)$$

$$T_s \frac{dx_s}{dt} = x_1 - (\bar{\mu}_2 + \beta_1 \beta_2) x_s +$$

$$+ \beta_1 (1 - \rho_1) \eta^-(t - \psi_c T_c) - (\pi_s^0 + x_s) u_2(t) \quad (21)$$

$$T_2 \frac{dx_2}{dt} = \bar{\mu}_2 x_s - x_2 + (\pi_s^0 + x_s) u_2(t)$$

$$\eta^+(t) = \rho_1 \eta^-(t - \psi_c T_c) + \beta_2 x_s(t)$$

$$\eta^-(t) = \rho_2 \eta^+(t - \psi_c T_c)$$

where the newly introduced parameters have the following properties:  $|\rho_1| < 1$ ,  $0 < \rho_2 < 1$ ,  $\beta_2 > 0$ .

In the following sections of the paper we shall focus on system (21).

### 3. ANALYSIS OF THE BASIC PROPERTIES FOR THE SYSTEM (21)

**A.** The basic theory - existence, uniqueness and data dependence - relies on the construction by steps of the solution for the difference part of (21). The initial conditions for (21) are obtained as follows, starting from the initial conditions of (15). Given  $s(0)$ ,  $x_k(0)$ ,  $k = 1, 2$ ,  $x_s(0)$ ,  $\zeta_\rho^0(\lambda)$ ,  $\zeta_w^0(\lambda)$  ( $0 \leq \lambda \leq 1$ ), the initial conditions  $\zeta_0^\pm(\lambda)$  for the Riemann invariants are obtained from (16). Further, we integrate  $\zeta^+(\lambda, t)$  along those increasing characteristics that cannot be extended up to  $\lambda = 0$  since they cross the axis  $t = 0$  before:  $t^+(\sigma; \lambda, t) = t + \psi_c T_c (\sigma - \lambda) = 0$  for  $\hat{\sigma} = \lambda - t/(\psi_c T_c) > 0$  provided  $t - \lambda \psi_c T_c < 0$ . Therefore

$$\begin{aligned} \zeta^+(1, t + \psi_c T_c (1 - \lambda)) &= \zeta^+(\lambda - t/(\psi_c T_c), 0), \\ 0 < \lambda - t/(\psi_c T_c) < 1 \end{aligned}$$

that is

$$\eta^+(\theta) = \zeta_0^+(-\theta/(\psi_c T_c)), \quad -\psi_c T_c \leq \theta \leq 0 \quad (22)$$

In a similar way we obtain

$$\zeta^-(0, t + \psi_c T_c \lambda) = \zeta^-(\lambda + t/(\psi_c T_c), 0), \quad 0 < \lambda + t/(\psi_c T_c) < 1$$

that is

$$\eta^-(\theta) = \zeta_0^-(1 + \theta/(\psi_c T_c)), \quad -\psi_c T_c \leq \theta \leq 0 \quad (23)$$

Given  $u_k(t)$  on some interval  $[0, T]$  and the aforementioned initial conditions, the construction by steps of the solution of (21) is obvious. The solution is unique and  $\eta^\pm(t)$  have the smoothness of the initial conditions and are discontinuous at  $t = k\psi_c T_c$  where  $k$  is a positive integer. We deduce that (21) is a system of neutral type with all subsequent properties; moreover its difference operator is strongly stable since  $|\rho_k| < 1$ ,  $k = 1, 2$ . Now, formulae (18) and (19) can be viewed as genuine representation formulae for (15) since

$$\zeta^+(\lambda, t) = \eta^+(t - \psi_c T_c \lambda), \quad \zeta^-(\lambda, t) = \eta^-(t + \psi_c T_c (\lambda - 1)) \quad (24)$$

and

$$\begin{aligned} \zeta_\rho(\lambda, t) &= \frac{1}{2}[\eta^+(t - \psi_c T_c \lambda) + \eta^-(t + \psi_c T_c (\lambda - 1))] \\ \zeta_w(\lambda, t) &= \frac{1}{2}[\eta^+(t - \psi_c T_c \lambda) - \eta^-(t + \psi_c T_c (\lambda - 1))] \end{aligned} \quad (25)$$

Therefore the set of functions  $(\zeta_\rho(\lambda, t), \zeta_w(\lambda, t), s(t), x_k(t), x_s(t))$  defines a classical solution for (15), with propagating discontinuities at  $t \pm \psi_c T_c \lambda = k\psi_c T_c$  where  $k$  is an integer. This classical solution is however continuous provided the initial conditions are ‘‘matched’’ to the boundary conditions

$$\begin{aligned} \psi_s \zeta_w^0(0) + (1 - \psi_s^2) \zeta_\rho^0(0) &= \sqrt{e^{-\psi_s^2}} x_s(0), \\ \zeta_w^0(1) &= \psi_s \zeta_\rho^0(1) \end{aligned} \quad (26)$$

(these are also called *consistency conditions*).

**B.** In order to obtain some invariant set accounting for positivity of some state variables, we make use of (14) in (15) and (21) to obtain

$$\psi_c T_c \partial_t \xi_\rho + \partial_\lambda \xi_w = 0, \quad \psi_c T_c \partial_t \xi_w + \partial_\lambda \xi_\rho = 0$$

$$\psi_s \xi_w(0, t) + (1 - \psi_s^2) \xi_\rho(0, t) = \sqrt{e^{-\psi_s^2}} x_s$$

$$\xi_w(1, t) = \psi_s \xi_\rho(1, t)$$

$$T_a \frac{ds}{dt} = \alpha \pi_1 + (1 - \alpha) \pi_2 - v_g, \quad T_1 \frac{d\pi_1}{dt} = -\pi_1 + \mu_1(t) \quad (27)$$

$$T_s \frac{d\pi_s}{dt} = \pi_1 - \mu_2(t) \pi_s - \beta_1 \xi_w(0, t)$$

$$T_2 \frac{d\pi_2}{dt} = \mu_2(t) \pi_s - \pi_2$$

and, subsequently

$$T_a \frac{ds}{dt} = \alpha \pi_1 + (1 - \alpha) \pi_2 - v_g, \quad T_1 \frac{d\pi_1}{dt} = -\pi_1 + \mu_1(t)$$

$$T_s \frac{d\pi_s}{dt} = \pi_1 - (\mu_2(t) + \beta_1 \beta_2) \pi_s + \beta_1 (1 - \rho_1) y^-(t - \psi_c T_c)$$

$$T_2 \frac{d\pi_2}{dt} = \mu_2(t) \pi_s - \pi_2$$

$$y^+(t) = \rho_1 y^-(t - \psi_c T_c) + \beta_2 \pi_s(t), \quad y^-(t) = \rho_2 y^+(t - \psi_c T_c) \quad (28)$$

where  $y^\pm(t) = \eta^\pm(t) + (\bar{\xi}_\rho \pm \bar{\xi}_w)$ .

To simplify the analysis assume that  $\rho_1 > 0$  i.e.  $1 - \psi_s - \psi_s^2 < 0$  ( $(\sqrt{5} - 1)/2 < \psi_s < 1$ ). Since  $\mu_k(t) > 0$ , if the initial conditions for  $\pi_k$ ,  $\pi_s$ ,  $y^\pm(t)$  are positive, the construction by steps will result in positive  $\pi_k(t)$ ,  $\pi_s(t)$ ,  $y^\pm(t)$ . Therefore the Riemann invariants are positive hence  $\xi_\rho(\lambda, t) > \xi_w(\lambda, t)$ . We have proved in fact the following *augmented validation* result

**Theorem 1.** Consider the system (27) and the associated system of functional equations (28) with positive parameters  $\psi_c T_c > 0$ ,  $(\sqrt{5} - 1)/2 < \psi_s < 1$ ,  $0 < \alpha < 1$ ,  $\beta_1 > 0$ ,  $\beta_2 > 0$ ,  $0 < \rho_k < 1$  and with the control signals subject to  $0 \leq \mu_1(t) \leq 1$ ,  $0 < \mu_{2min} \leq \mu_2(t) \leq 1$ . Then (28) and (27) subsequently have a unique solution, possibly discontinuous at  $k\psi_c T_c$  and  $t \pm \psi_c T_c \lambda = k\psi_c T_c$  respectively, with the smoothness of their initial conditions. Moreover, these systems display the invariant sets

$$\{\pi_1 > 0, \pi_s > 0, \pi_2 > 0, y^\pm(\cdot) > 0\}$$

and

$$\{\pi_1 > 0, \pi_s > 0, \pi_2 > 0, \xi_\rho(\cdot) > 0, \xi_\rho(\cdot) > \xi_w(\cdot)\}$$

respectively.

**C.** We discuss now the inherent stability of the equilibria as suggested by the Stability Postulate of N.G. Četaev. Consider system (21) with  $\mu_k(t) \equiv 0$ ,  $k = 1, 2$ . The system is now linear and has the following characteristic equation (obtained after some simple and straightforward manipulation)

$$\begin{aligned} T_a \sigma (T_1 \sigma + 1) (T_2 \sigma + 1) (p_c(\sigma) \cosh \sigma \psi_c T_c + \\ + p_s(\sigma) \sinh \sigma \psi_c T_c) = 0 \end{aligned} \quad (29)$$

where we denoted

$$p_c(\sigma) := (1 - \rho_1 \rho_2) T_s \sigma + \bar{\mu}_2 (1 - \rho_1 \rho_2) + \beta_1 \beta_2 (1 - \rho_2)$$

$$p_s(\sigma) := (1 + \rho_1 \rho_2) T_s \sigma + \bar{\mu}_2 (1 + \rho_1 \rho_2) + \beta_1 \beta_2 (1 + \rho_2)$$

Equation (29) has a simple zero root and all other factors have roots in  $\mathbb{C}^-$  (for the first two is obvious, for the quasi-polynomial, see Čebotarev and Meiman (1949), Chapter VII).

Taking into account the engineering requirements for asymptotic stability, system (21) needs feedback stabilization.

#### 4. A CONTROL LYAPUNOV FUNCTIONAL AND SYNTHESIS OF THE FEEDBACK CONTROL

We shall consider again system (21) which has inherent stability but non- asymptotic stability. Considering, as it is the case in Power Control Engineering, that  $T_1$  and  $T_2$  - the time constant of the turbine cylinders - are small time constants, we take  $T_1 = T_2 = 0$  in (21) to obtain the following system with reduced dynamics

$$\begin{aligned} T_a \frac{ds}{dt} &= (1 - \alpha)\bar{\mu}_2 x_s + \alpha u_1(t) + (1 - \alpha)(\pi_s^0 + x_s)u_2(t) \\ T_s \frac{dx_s}{dt} &= -(\bar{\mu}_2 + \beta_1 \beta_2)x_s + \beta_1(1 - \rho_1)\rho_2 \eta^+(t - 2\psi_c T_c) + \\ &\quad + u_1(t) - (\pi_s^0 + x_s)u_2(t) \\ \eta^+(t) &= \rho_1 \rho_2 \eta^+(t - 2\psi_c T_c) + \beta_2 x_s(t) \end{aligned} \quad (30)$$

(For stability and stabilization studies the equation of  $\eta^-(t)$  can be eliminated since it appears as a system output).

To system (30) we associate the following quadratic Lyapunov functional

$$\begin{aligned} \mathcal{V}(s, x_s, \phi(\cdot)) &= \frac{1}{2} T_a \left[ s + \frac{T_s}{T_a} \left( \delta_1 x_s + \frac{\delta_2}{T_s} \int_{-2\psi_c T_c}^0 \phi(\theta) d\theta \right) \right]^2 + \\ &\quad + \frac{1}{2} \delta_3 T_s x_s^2 + \delta_4 \int_{-2\psi_c T_c}^0 \phi(\theta)^2 d\theta \end{aligned} \quad (31)$$

Along the solutions of (30)  $\phi(\cdot)$  is  $\eta_t^+(\cdot)$ . Differentiating  $\mathcal{V}(s(t), x_s(t), \eta_t^+(\cdot))$  along (30) the following derivative functional is obtained

$$\begin{aligned} \mathcal{W}(s, x_s, \phi(\cdot)) &= [(\alpha + \delta_1)\mathcal{L}(s, x_s, \phi(\cdot)) + \delta_3 x_s] u_1 + \\ &\quad + (\pi_s^0 + x_s)[(1 - \alpha - \delta_1)\mathcal{L}(s, x_s, \phi(\cdot)) - \delta_3 x_s] u_2 + \\ &\quad + \mathcal{L}(s, x_s, \phi(\cdot))[(1 - \alpha)\bar{\mu}_2 x_s - \delta_1(\bar{\mu}_2 + \beta_1 \beta_2)x_s + \\ &\quad + \delta_1 \beta_1 \rho_2 (1 - \rho_1)\phi(-2\psi_c T_c) + \\ &\quad + \delta_2(\phi(0) - \phi(-2\psi_c T_c))] - \mathcal{Q}(x_s, \phi(\cdot)) \end{aligned} \quad (32)$$

where the linear form  $\mathcal{L}(s, x_s, \phi(\cdot))$  and the quadratic form  $\mathcal{Q}(x_s, \phi(\cdot))$  are given by

$$\begin{aligned} \mathcal{L}(s, x_s, \phi(\cdot)) &= s + \frac{T_s}{T_a} \left( \delta_1 x_s + \frac{\delta_2}{T_s} \int_{-2\psi_c T_c}^0 \phi(\theta) d\theta \right) \\ \mathcal{Q}(s, \phi(\cdot)) &= \delta_3(\bar{\mu}_2 + \beta_1 \beta_2)x_s^2 - \\ &\quad - \delta_3 \beta_1 \rho_2 (1 - \rho_1)x_s \phi(-2\psi_c T_c) + \delta_4(\phi(0)^2 - \phi(-2\psi_c T_c)^2) \end{aligned} \quad (33)$$

The first choice goes for the control signals as feedback state functions

$$\begin{aligned} u_1 &= -\text{Sat}[(\alpha + \delta_1)\mathcal{L}(s, x_s, \phi(\cdot)) + \delta_3 x_s], \\ u_2 &= -\text{Sat}[(1 - \alpha - \delta_1)\mathcal{L}(s, x_s, \phi(\cdot)) - \delta_3 x_s] \end{aligned} \quad (34)$$

with  $-\bar{\mu}_1 \leq u_1 \leq 1 - \bar{\mu}_1$  and  $\mu_{2min} - \bar{\mu}_2 \leq u_2 \leq 1 - \bar{\mu}_2$ .

The next choice concerns the free parameters  $\delta_1$  and  $\delta_2$ : they are chosen in order to eliminate the term linear in  $\mathcal{L}(s, x_s, \phi(\cdot))$ . Making use of the equation of  $\eta^+(t)$  we obtain

$$1 - \alpha < \delta_1 = \frac{(1 - \rho_1 \rho_2)(1 - \alpha)\bar{\mu}_2}{(1 - \rho_1 \rho_2)\bar{\mu}_2 + (1 - \rho_2)\beta_1 \beta_2} < 1 \quad (35)$$

$$\delta_2 = \frac{\beta_1 \rho_2 (1 - \rho_1)}{1 - \rho_1 \rho_2} \delta_1 = \frac{(1 - \alpha)\beta_1 \rho_2 (1 - \rho_1)\bar{\mu}_2}{(1 - \rho_1 \rho_2)\bar{\mu}_2 + (1 - \rho_2)\beta_1 \beta_2}$$

The third choice concerns the free parameters  $\delta_3 > 0$  and  $\delta_4 > 0$ : they are chosen in order to make the quadratic form  $\mathcal{Q}(s, \phi(\cdot))$  positive definite. Making again use of the difference equation in (30) we find that the ratio  $\delta_3/\delta_4 > 0$  has to be chosen between the two positive roots of the trinomial

$$\begin{aligned} \mathcal{T}(X) &= \beta_1^2 \rho_2^2 (1 - \rho_1)^2 X^2 - \\ &\quad - 4(\bar{\mu}_2(1 - \rho_1^2 \rho_2^2) + \beta_1 \beta_2(1 - \rho_1 \rho_2^2))X + 4\beta_2^2(1 - \rho_1^2 \rho_2^2) \end{aligned} \quad (36)$$

With these choices of the parameters the derivative functions becomes

$$\begin{aligned} \mathcal{W}(s, x_s, \phi(\cdot)) &= -[(\alpha + \delta_1)\mathcal{L}(s, x_s, \phi(\cdot)) + \delta_3 x_s] \times \\ &\quad \times \text{Sat}[(\alpha + \delta_1)\mathcal{L}(s, x_s, \phi(\cdot)) + \delta_3 x_s] - \\ &\quad - (\pi_s^0 + x_s)[(1 - \alpha - \delta_1)\mathcal{L}(s, x_s, \phi(\cdot)) - \delta_3 x_s] \times \\ &\quad \times \text{Sat}[(1 - \alpha - \delta_1)\mathcal{L}(s, x_s, \phi(\cdot)) - \delta_3 x_s] - \\ &\quad - \mathcal{Q}(x_s, \phi(-2\psi_c T_c)) \leq 0 \end{aligned} \quad (37)$$

From (37) we deduce that

$$\mathcal{V}(s(t), x_s(t), \eta_t^+(\cdot)) \leq \mathcal{V}(s(0), x_s(0), \eta_0^+(\cdot)) \quad (38)$$

which signifies uniform stability in the sense of the norm induced by the Lyapunov functional  $\mathcal{V}$ . For the asymptotic stability we remark that  $\mathcal{W} = 0$  on the set where  $x_s = 0$ ,  $\phi(-2\psi_c T_c) = 0$ . Applying the Barbashin Krasovskii LaSalle invariance principle for neutral equations (Theorem 8.2 of Section 9.8 in Hale and Lunel (1993)), global asymptotic stability follows.

Making use of the representation formulae (24) and (25), we obtained in fact global asymptotic stability for the system below, deduced from (15) by letting  $T_1 = T_2 = 0$

$$\psi_c T_c \partial_t \zeta_\rho + \partial_\lambda \zeta_w = 0, \quad \psi_c T_c \partial_t \zeta_w + \partial_\lambda \zeta_\rho = 0$$

$$\psi_s \zeta_w(0, t) + (1 - \psi_s^2) \zeta_\rho(0, t) = \sqrt{e^{-\psi_s^2 t}} x_s$$

$$\zeta_w(1, t) = \psi_s \zeta_\rho(1, t) \quad (39)$$

$$T_a \frac{ds}{dt} = (1 - \alpha)\bar{\mu}_2 x_s + \alpha u_1(t) + (1 - \alpha)(\pi_s^0 + x_s)u_2(t)$$

$$T_s \frac{dx_s}{dt} = -\bar{\mu}_2 x_s - \beta_1 \zeta_w(0, t) + u_1(t) - (\pi_s^0 + x_s)u_2(t)$$

with  $u_1$  and  $u_2$  chosen from (33) and with the integral expressed from the representation formulae (24) and (25). Summarizing we have obtained the following

*Theorem 2.* Consider the system (30) with all parameters positive. If the control functions  $u_k$  are chosen according to (34) with  $\delta_1 > 0$ ,  $\delta_2 > 0$  taken from (35), the zero solution of (30) is globally asymptotically stable. Consequently, the zero solution of (39) is also globally asymptotically stable provided  $\mathcal{L}(s, x_s, \phi(\cdot))$  is replaced by the linear state functional

$$\begin{aligned} \mathcal{L}(s, x_s, \phi_\rho(\cdot), \phi_w(\cdot)) &= s + \frac{T_s}{T_a} (\delta_1 x_s + \\ &\quad + \frac{\delta_2}{T_s} \cdot \frac{2\psi_c T_c}{1 - \psi_s} \int_0^1 (\phi_\rho(\lambda) - \psi_s \phi_w(\lambda)) d\lambda) \end{aligned} \quad (40)$$

Let us observe that the structure of the feedback - linear saturated - is standard in Power Control Engineering. Normally the control feedback for bilinear systems results quadratic - see e.g. Slemrod (1978); Quinn (1980) - but the invariant set  $\pi_s > 0$  i.e.  $\pi_s^0 + x_s > 0$  allowed obtaining a linear one. An additional remark connected to the distributed parameters is the following: if the feedback with respect to  $s$  and  $x_s$  is standard in steam turbine control, the integral - “averaging” - term is specific to our approach but not unusual in distributed parameters. Simulation studies show that if this term is missing the system might not be stabilized - see Danciu et al. (2015).

## 5. CONCLUDING REMARKS. PERSPECTIVE PROBLEMS

In this paper we started from a nonlinear system of conservation laws with nonlinear boundary conditions that has been linearized in two stages. It resulted finally a bilinear controlled system of functional differential equations, stabilized by a saturated linear feedback control. The closed loop system is nonlinear with a globally asymptotically stable equilibrium at the origin. This property is projected back on the zero solution for the closed loop system with distributed parameters which is also globally asymptotically stable.

The engineering significance of the results and their practical importance are obvious from the physical significance of the equations. For this reasons (but not only) it is of interest to enumerate some open problems which can define an interesting research program consisting of: **a)** preservation of the global asymptotic stability face to the neglected small time constants; this can be analyzed using techniques of singular perturbations; **b)** proof of the stability by the first approximation for the nonlinear boundary value problem; such a mathematical result is strongly dependent of the theorem of K.P. Persidskii type in the nonlinear case, ensuring that global asymptotic stability implies exponential stability; such a theorem has been proved by A. Halanay for ordinary differential equations and in Răsvan (2012) for time delay equations; for neutral equations this theorem is not known. **c)** development of an augmented validation theory (again in the sense of Răsvan (2014)) for systems containing nonlinear difference operators.

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