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Stability of time-periodic and delayed systems — a route to act-and-wait control

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Abstract

The history of the linear time-delayed and time-periodic oscillator demonstrates how stability theory has developed from the damped oscillator to the delayed Mathieu equation. Based on these results, it is a natural idea to apply time-periodic control gains when large time delay in the feedback loop tends to destabilize the system. By formulating the act-and-wait control concept as a special case of periodic controllers, a time delayed version of the Brockett problem is posed. Examples demonstrate the efficiency of the kind of time periodic control where the feedback is switched off for a waiting interval longer than the delay.

Keywords: Time delay; Stability; Act-and-wait; Periodic control; Oscillator

1. Introduction

The stabilization of dynamical systems with large time delay is still a challenging task. The first engineering applications where time delay played a key role were the shimmying wheels with elastic tyres (von Schlippe and Dietrich, 1941), the famous ship stabilization problem (Minorsky, 1942), and the material forming processes (Tlusty, Polacek, Danek, & Spacek, 1962; Tobias, 1965). The development of the corresponding mathematical theory of time delayed systems started in the early 1950's only (Bellman & Cooke, 1963; Diekmann, van Gils, Lunel, & Walther, 1995; Halanay, 1961; Hale & Lunel, 1993; Myshkis, 1949). The first algorithms for stability analyses appeared somewhat later in the work of Bhatt and Hsu (1966); Kolmanovskii and Nosov (1986); Stépán (1989); Niculescu (2000) and Hu and Wang (2002).

The mathematical description of parametric excitation goes back to the mid 19th century, when Mathieu (1868) derived his famous scalar second-order time-periodic differential equation to study the vibrations of elliptic membranes. Although the theoretical basis for stability analysis in periodic systems was provided by the Floquet Theory in the 19th century (Floquet, 1883), the first stability results appeared in the literature only decades later, like the stabilization of the inverted pendulum by vibrating its pivot point vertically at a specific frequency (Stephenson, 1908), or the stability chart of the Mathieu equation (Ince, 1956; van der Pol & Strutt, 1928). The explanation of how the children's favourite toy, the swing works, came also quite late for the same reason (Levi & Broer, 1995).

Stability of periodic systems are characterized by the Floquet transition matrix and by its eigenvalues, the characteristic multipliers (often referred to as poles of the system). If all of the characteristic multipliers lies in the open unit disc of the complex plane then the system is asymptotically stable (see, e.g., Farkas, 1994).

Nowadays, as both autonomous delayed systems and parametrically excited systems are quite well understood, those engineering models show up and come into focus, where the two effects may exist together. In this respect, one of the most apparent engineering problems is high-speed milling, where the time delay is caused by the regeneration of the wavy surface cut by the compliant tool, and parametric excitation arises due to the rotation of the tool (Budak & Altintas, 1998; Insperger, Mann, Stépán, & Bayly, 2003a). In the milling processes, both the time delay and the principal period are equal to the tooth passing period. Another area of engineering where the combination of time delay and parametric excitation shows up is the control of dynamical systems (Butcher, Ma, Bueler,

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Averina, & Szabó, 2004; Elbeyly & Sun, 2004; Insperger & Stépán, 2000).

Time delay often arises in feedback control systems due to the acquisition of response and excitation data, information transmission, on-line data processing, computation and application of control forces. In spite of the efforts to minimize time delays, they cannot be eliminated totally even with today's advanced technology, due to physical limits. The information delay is often negligible but, for some cases, it may still be crucial, for example in space applications (Kim & Bejczy, 1993; Vertut, Charles, Coiffet, & Petit, 1976), in systems controlled through the internet (Munir & Book, 2003) or in robotic applications with time-consuming control force computation (Kovács, Insperger, & Stépán, 2004).

Caused by the delay of control feedback, the governing equation is a delay-differential equation (DDE). DDE's usually have infinite dimensional phase spaces (Hale & Lunel, 1993), therefore the linear stability conditions for the system parameters are complicated and often do not have an analytical form. However, there exist several methods to analyze control systems with delayed feedback (see, e.g., Breda, Maset, & Vermiglio, 2004; Butcher et al., 2004; Insperger & Stépán, 2002a; Michiels, Engelborghs, Vansevenant, & Roose, 2002; Olgac & Sipahi, 2002).

Stability of time-periodic DDEs are described by the so-called monodromy operator that corresponds to an infinite dimensional Floquet transition matrix with infinite number of eigenvalues (characteristic multipliers or poles). The system is asymptotically stable if all the characteristic multipliers are in modulus less than one (Hale & Lunel, 1993).

Although time-invariant state feedback is a wide-spread and easily applicable technique for control systems, it does not provide a stabilizing controller for all systems. In these cases, the use of time-periodic feedback gains may improve stability properties. The idea of stabilizing by parametric excitation is motivated by the classical example of Stephenson's inverted pendulum (Stephenson, 1908).

The problem of stabilization by means of time-periodic feedback gains has been presented by Brockett (1998) as one of the challenging open problems in control theory. With the exception of some papers on discrete-time systems (Aeyels & Willems, 1992; Leonov, 2002a; Weiss, 2005), the problem has received little attention and only partial results for special classes of systems have been derived. Moreau and Aeyels (1999, 2000) investigated the effect of sinusoidal feedback gain for second and third order systems. Leonov (2002b) and Allwright, Astolfi, and Wong (2005) used piecewise constant control gains for general n-dimensional systems.

Stabilization is a weak version of pole placing. Stabilization of autonomous systems via time-invariant feedback control is the placement of the system's characteristic roots (called also characteristic exponents or poles) to the left half of the complex plane. If time-periodic gains are used in the control, then stabilization means the placement of the characteristic multipliers of the system inside the unit circle of the complex plane in accordance with the Floquet theory (see, e.g., Farkas, 1994). Details will be given in Section 3.

Stabilization of time delayed systems is complicated, since an infinite number of poles should be controlled using a finite number of control parameters. The act-and-wait control method is an effective technique to reduce the number of poles for systems with large feedback delay, which makes the pole placement problem easier. It is a special case of periodic controllers: the controller is switched on and off periodically with a switch off period larger than the time delay. The method was introduced for discrete-time systems in Insperger and Stépán (2004, 2006), where it was shown that for certain conditions, deadbeat control can also be achieved. The act-and-wait concept was adopted for continuous-time systems in Insperger (2006).

In this paper, we summarize the analytical results known for second-order systems, from the autonomous non-delayed system to the periodic delayed one including the delayed Mathieu equation. Then, inspired by these results, we introduce the act-and-wait control concept in a general form for continuous-time systems. Finally, the efficiency of the method is demonstrated by case studies.

2. Motivating example: the delayed Mathieu equation

The delayed Mathieu equation is a paradigm for periodic time delayed systems. It is one of the simplest equations that incorporates both delayed feedback effect and parametric excitation and still has practical relevance. Here, we consider the delayed Mathieu equation with slight damping in the scalar form:

$$\ddot{x}(t) + \kappa\dot{x}(t) + (\delta + \varepsilon\cos t)x(t) = bx(t - 2\pi). \tag{1}$$

In this example, the time delay is normalized to 2π with an appropriate time scale transformation, and it is just equal to the time period of the stiffness parameter in the same way as in case of milling processes (see, e.g., Insperger et al., 2003). The scalar parameter κ is the (small) damping coefficient in the system, the parameters δ and ε are the mean and the amplitude of the harmonic stiffness variation, and the parameter b is the gain of the delayed feedback.

When the stiffness amplitude and the gain are zero, the classical damped oscillator is described by

$$\varepsilon = 0, \quad b = 0 \Rightarrow \ddot{x}(t) + \kappa\dot{x}(t) + \delta x(t) = 0. \tag{2}$$

According to the basic theory of linear autonomous ordinary differential equations (ODEs) (proposed by Maxwell in 1865), the zeros of the characteristic polynomial determine the stability properties of (2): if and only if all the characteristic roots have negative real parts then the system is asymptotically stable. The famous Routh–Hurwitz criterion provides an algorithm to check this condition in characteristic polynomials (see Hurwitz, 1895; Routh, 1877). Accordingly, (2) has asymptotically stable trivial solutions if and only if $\kappa > 0$ and $\delta > 0$. The corresponding stability chart has the trivial structure given in Fig. 1; this is the first of a series of charts leading to that of the delayed Mathieu equation.

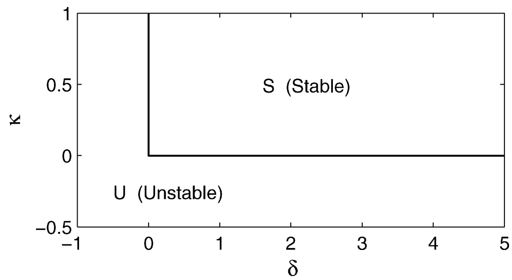

Fig. 1. Stability boundaries for the damped oscillator (2).

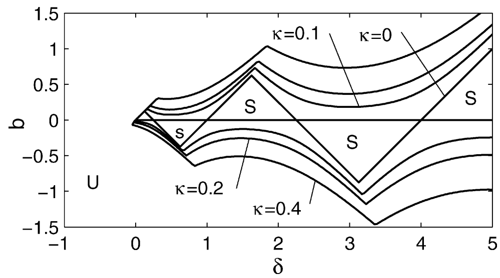

Fig. 3. Stability boundaries for the delayed oscillator (4).

In the uncontrolled case, we obtain the damped Mathieu equation (Mathieu, 1868):

$$b = 0 \Rightarrow \ddot{x}(t) + \kappa\dot{x}(t) + (\delta + \varepsilon\cos t)x(t) = 0. \tag{3}$$

Although the so-called Hill's infinite determinant method was available in the literature (Hill, 1886; Rayleigh, 1887), van der Pol and Strutt (1928) published the corresponding stability chart (often referred to as Strutt–Ince chart) in analytical form only much later. The chart is shown in Fig. 2 for different damping parameters. As it is seen, the system can be stable even for negative values of δ, a situation that corresponds to the inverted pendulum stabilized by parametric excitation (Stephenson, 1908).

In the case of an oscillator subjected to delayed feedback without parametric excitation, we have the so-called delayed oscillator equation:

$$\varepsilon = 0 \Rightarrow \ddot{x}(t) + \kappa\dot{x}(t) + \delta x(t) = bx(t - \tau). \tag{4}$$

With the help of the D-subdivision method (Neimark, 1949), one can prove that for the undamped case ($\kappa = 0$), the stability boundaries are straight lines in the parameter plane (δ, b). To select the stable domains among them, Bhatt and Hsu (1966) applied the method of Pontryagin (1942). Since then, more general stability criteria have appeared in the literature, like those of Stépán (1989) or Olgac and Sipahi (2002). The corresponding chart is presented in Fig. 3.

The early paper of Halanay (1961), started the development of the infinite dimensional version of the Floquet Theory for

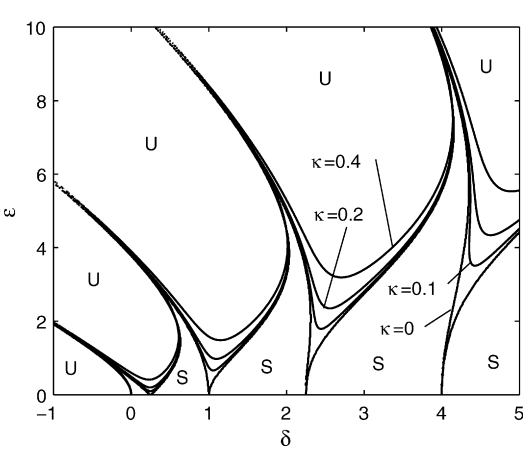

Fig. 2. Stability boundaries for the Mathieu Eq. (3).

delayed systems, but it gave only a theoretical possibility for the stability analysis of the most general case of the delayed Mathieu Eq. (1). After several attempts to work out an algorithm for the stability analysis of time-periodic time-delayed systems, it was the straight generalization of the classical Hill's infinite determinant method that provided an analytical solution to the problem (see Insperger and Stépán, 2002b).

In the undamped case, the delayed oscillator subjected to harmonic parametric excitation assumes the form:

$$\kappa = 0 \Rightarrow \ddot{x}(t) + (\delta + \varepsilon\cos t)x(t) = bx(t - \tau). \tag{5}$$

In Insperger and Stépán (2002b), it was proven that the stability boundaries remain straight lines in the parameter plane (δ, b) for any fixed value of ε, and these lines are passing along the $\kappa = 0$ boundary curves of the Strutt–Ince diagram of Fig. 2 for varying parametric excitation amplitude $\varepsilon > 0$, as it is shown in Fig. 4. The stability boundaries in the parameter plane (δ, ε) are shown in Fig. 5, where continuous lines denote stability boundaries for $b \geq 0$ and dashed lines denote stability boundaries for $b < 0$. The three-dimensional representation of the stability chart is shown in Fig. 6.

With a further generalization of the above methods and results, the straight-line boundaries were also found and proven in the most general case of the delayed Mathieu Eq. (1). The non-zero damping merges the triangle shaped stable domains of the delayed oscillator, but the parametric excitation cuts these regions by some separating lines at fixed values $\varepsilon > 0$ of the excitation amplitude (see Insperger and Stépán (2003)). A typical stability chart of (1) is presented in Fig. 7 for $\varepsilon = 2$.

It rarely happens, of course, that the above charts can directly be transformed to the parameter space of realistic

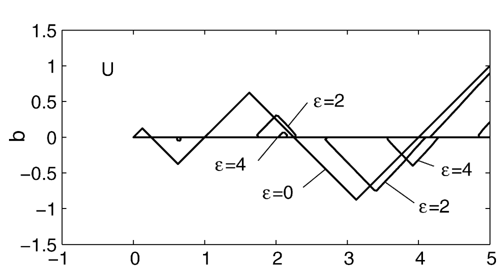

Fig. 4. Stability boundaries for the undamped delayed Mathieu Eq. (5) in the plane (δ, b).

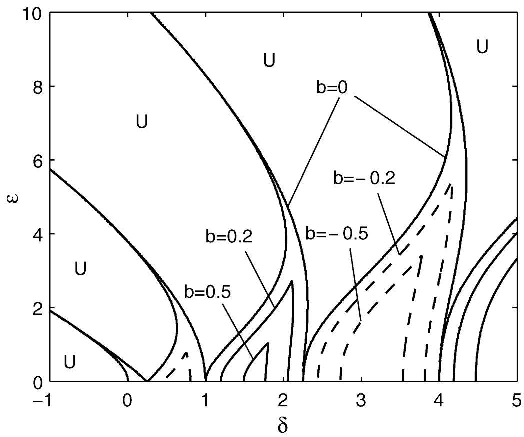

Fig. 5. Stability boundaries for the undamped delayed Mathieu Eq. (5) in the plane (δ, ε).

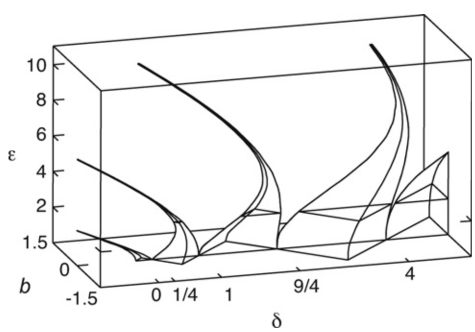

Fig. 6. Three-dimensional stability chart for the undamped delayed Mathieu Eq. (5).

physical problems, but they serve as unique and reliable reference examples to test numerical methods, and they also provide an overall picture, helping to understand the peculiar stability behaviour of these time-periodic and also time-delayed second order systems.

It is an obvious extension of the above results to consider the periodicity at the gain parameters:

$$\ddot{x}(t) + \kappa\dot{x}(t) + \delta x(t) = (b + \varepsilon\cos t)x(t - \tau). \tag{6}$$

The study of this system with different time-periodic gains leads to the idea of the act-and-wait control concept that is introduced and discussed in the next section.

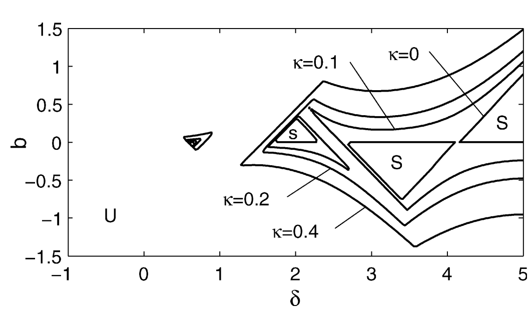

Fig. 7. Stability boundaries of the damped and delayed Mathieu Eq. (1) for $\varepsilon = 2$.

3. Act-and-wait control concept

Consider the control system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(t) \in \mathbb{R}^n, \quad \mathbf{u}(t) \in \mathbb{R}^m, \tag{7}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \mathbf{y}(t) \in \mathbb{R}^l \tag{8}$$

with the delayed feedback controller:

$$\mathbf{u}(t) = \mathbf{D}\mathbf{y}(t - \tau), \tag{9}$$

where τ is the time delay of the feedback loop. We assume that the delay is a fixed parameter of the feedback and cannot be eliminated or tuned during the control design.

System (7) and (8) with controller (9) implies the DDE:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{BDC}\mathbf{x}(t - \tau). \tag{10}$$

Stabilization of this system by tuning the control parameters in **D** in an optimal way is a difficult task since the corresponding characteristic equation:

$$\det(\lambda\mathbf{I} - \mathbf{A} - \mathbf{BDC}\,e^{-\tau\lambda}) = 0 \tag{11}$$

has infinitely many roots, which all should be placed in the left half of the complex plane in order to obtain asymptotic stability.

Arbitrary pole placement for (10) is not possible, since infinitely many poles should be placed using a finite number of control parameters (i.e., the elements of **D**). If the pair $(\mathbf{A}, \mathbf{B})$ is controllable then direct placement of n different eigenvalues is always possible. However, by placing n eigenvalues, the control over the position of all the others is lost, and these may cause instability (see, e.g., Michiels et al., 2002).

An effective technique to reduce the number of poles for delayed systems is the act-and-wait method (Insperger, 2006; Insperger & Stépán, 2004; Insperger & Stépán, 2006). The essence of the method is that the controller is switched on and off periodically with a switch-off interval longer than the time delay. This way, the memory effect of the feedback is eliminated, and a finite dimensional system is obtained instead of the infinite dimensional one.

The act-and-wait controller for system (7) and (8) can be defined as

$$\mathbf{u}(t) = \mathbf{G}(t)\mathbf{y}(t - \tau), \tag{12}$$

where $\mathbf{G}(t)$ is the T-periodic act-and-wait matrix function satisfying:

$$\mathbf{G}(t) = \begin{cases} 0 & \text{if } 0 \le t < t_w \\ \mathbf{\Gamma}(t) & \text{if } t_w \le t < t_w + t_a = T \end{cases} \tag{13}$$

and $\mathbf{G}(t + kT) = \mathbf{G}(t), k \in \mathbb{Z}$. Function $\mathbf{\Gamma}(t) : [t_w, T] \to \mathbb{R}^{m \times l}$ is an integrable matrix function. Using controller (12) instead of (9), the delayed feedback term is switched off for an interval of length t_w (wait), and it is switched on for an interval of length t_a (act). This is a special case of periodic controllers. Note that in Insperger (2006), the function $\mathbf{\Gamma}(t)$ was switched between zero and constant that corresponds to $\mathbf{\Gamma}(t) \equiv \mathbf{D}$.

Now, system (7) and (8) with controller (12) implies the time-periodic delay-differential equation (DDE):

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{G}(t)\mathbf{C}\mathbf{x}(t-\tau). \quad (14)$$

In the case $t_w < \tau$, (14) has an infinite number of characteristic multipliers, therefore stabilization is still rather complicated (if possible at all), similarly to the time-independent case (10). However, if $t_w \geq \tau$ then the monodromy operator of (14) becomes finite dimensional and can be represented by an $n \times n$ matrix as it is shown below.

Without loss of generality, we derive the solution map for $t \in [0, T]$ instead of $t \in [kT, (k+1)T]$, $k \in \mathbb{Z}$. First, assume that $t_w \geq \tau$ and $0 < t_a \leq \tau$. In this case, (14) can be considered as an ordinary differential equation (ODE) in $[0, t_w)$ and as a DDE in $[t_w, T)$. If $t \in [0, t_w)$ then $\mathbf{G}(t) = 0$ (the delayed term is switched off), and the solution of (14) associated with the initial state $\mathbf{x}(0)$ can be given as

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{x}(0), \qquad t \in [0, t_w). \quad (15)$$

If $t \in [t_w, T)$ then $\mathbf{G}(t) = \mathbf{\Gamma}(t)$ (the delayed term is switched on). Since $t_a \leq \tau$, and the solution over the interval $[0, t_w)$ is already given by (15), system (14) can be written in the form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{\Gamma}(t)\mathbf{C}\,\mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{x}(0), \quad t \in [t_w, T). \quad (16)$$

Solving (16) as a non-homogeneous ODE over $[t_w, T)$ with $\mathbf{x}(t_w) = \mathrm{e}^{\mathbf{A}t_w}\mathbf{x}(0)$ as initial condition we obtain:

$$\mathbf{x}(T) = \underbrace{\left(\mathrm{e}^{\mathbf{A}T} + \int_{t_w}^{T} \mathrm{e}^{\mathbf{A}(T-s)}\mathbf{B}\mathbf{\Gamma}(t)\mathbf{C}\,\mathrm{e}^{\mathbf{A}(s-\tau)}\,\mathrm{d}s\right)}_{\mathbf{\Phi}}\mathbf{x}(0). \quad (17)$$

This way, we constructed an $n \times n$ discrete map over the period T for the initial state $\mathbf{x}(0)$ using the piecewise solutions of (14). This means that the monodromy operator of (14) has n nonzero eigenvalues that are equal to the eigenvalues of the transition matrix $\mathbf{\Phi}$. All the other (infinitely many) eigenvalues of the monodromy operator are zero. Consequently, system (14) has n characteristic multipliers that are equal to the eigenvalues of matrix $\mathbf{\Phi}$ that is actually the monodromy matrix of the system.

Consider now the case when the period of acting cannot be smaller than the time delay due to any reason (e.g. physical limitations of the controller), that is, $t_w \geq \tau$ and $t_a > \tau$. In this case, the monodromy matrix can be determined by stepwise integration over subsequent intervals. First, the solution over $[0, t_w)$ should be determined similarly to (15). Then, the solutions can be determined over the intervals $[t_w, t_w + \tau)$, $[t_w + \tau, t_w + 2\tau)$, etc., step by step substituting the solution of the previous interval into the delayed term. If $h\tau < t_a \leq (h+1)\tau$, $h \in \mathbb{Z}$, then $\mathbf{x}(T)$ is obtained after $h+2$ succeeding integrations.

The point is that the act-and-wait method results in an $n \times n$ monodromy matrix, and only n poles should be considered during stabilization instead of the infinitely many poles of the original autonomous DDE (10). The control parameters are the elements of the matrix $\mathbf{\Gamma}(t)$. In this sense, the resulting problem is similar to the one posed by Brockett (1998) for ODEs. Here, it is formulated as:

Problem 1. For given matrices **A**, **B**, **C** and for given time delay τ, under what circumstances does there exist a time-varying matrix $\mathbf{\Gamma}(t)$ such that system (14) is asymptotically stable, i.e., all the eigenvalues of the monodromy matrix $\mathbf{\Phi}$ lie in the open unit disc of the complex plane?

This problem is more complex than that of Brockett since matrix $\mathbf{\Phi}$ depends nonlinearly on the system matrix **A** through the time delay τ. This problem has not been solved generally, but some case studies are given to demonstrate that an appropriate choice of $\mathbf{\Gamma}(t)$ can stabilize unstable systems.

4. Case studies

In this section, two examples will be presented to demonstrate the efficiency of the act-and-wait control concept: a first-order system, then the delayed oscillator as a second-order system.

4.1. First-order system

Consider the scalar DDE:

$$\dot{x}(t) = ax(t) + bx(t-1), \quad (18)$$

where a is the system parameter and b is the control gain. This equation corresponds to system (7)–(9) with the matrices:

$$\mathbf{A} = a, \quad \mathbf{B} = 1, \quad \mathbf{C} = 1, \quad \mathbf{D} = b \quad (19)$$

and with the delay $\tau = 1$. (Note that $n = m = l = 1$ in this case.)

The stability chart of (18) was first presented in Hayes (1950) (see in Fig. 8). This equation is often considered as one of the simplest basic examples for a delayed system (see, e.g., Michiels et al., 2002 or Stépán, 1989). Its stability properties

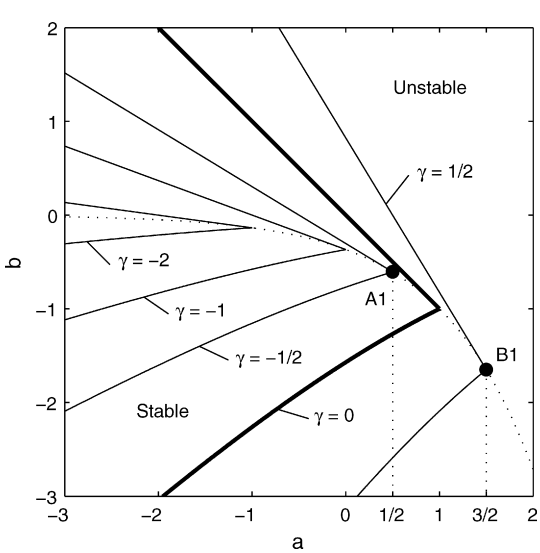

Fig. 8. Stability chart for (18)

can be determined by the analysis of the characteristic equation:

$$\lambda - a - b\,\mathrm{e}^{-\lambda} = 0. \tag{20}$$

Substitution of $\lambda = \gamma \pm i\omega$ into (20) provides the γ-contour curves:

$$a = \gamma + \frac{\omega\cos\omega}{\sin\omega}, \quad b = \frac{-\omega\,\mathrm{e}^{\gamma}}{\sin\omega} \tag{21}$$

for $\omega > 0$ and the lines $b = (\gamma - a)\mathrm{e}^{\gamma}$ for $\omega = 0$. In Fig. 8, some of these γ-contours are also presented. Obviously, the stability boundary corresponds to $\gamma = 0$ that is denoted by thick lines.

For a given system parameter a, the optimal control gain that results in minimal γ is given by $b = -\mathrm{e}^{a-1}$ associated with $\gamma = a - 1$. This curve is presented by a dotted line in Fig. 8. It can be seen that if $a > 1$ then the system is unstable for all b.

For further analysis, we introduce the decay ratio $\rho = \mathrm{e}^{\gamma}$, which is a measure of the average error decay over a unit period, since $|x(t+1)| \le \rho|x(t)|$.

Here, we will investigate two cases:

$a = 1/2$: the corresponding optimal decay ratio is $\rho = \mathrm{e}^{-1/2} = 0.6065 < 1$ (stable case) that can be achieved at $b = -\mathrm{e}^{-1/2} = -0.6065$. This case is denoted by point A1 in Fig. 8.

$a = 3/2$: the corresponding optimal decay ratio is $\rho = \mathrm{e}^{1/2} = 1.6487 > 1$ (unstable case) that can be achieved at $b = -\mathrm{e}^{1/2} = -1.6487$. This case is denoted by point B1 in Fig. 8.

Now, apply the act and wait concept with $\boldsymbol{\Gamma}(t) \equiv b$, $t_{\mathrm{w}} = 1$, $t_{\mathrm{a}} = 1$, $T = 2$ in (13). The governing DDE can be given in the form:

$$\dot{x}(t) = ax(t) + G(t)x(t-1), \tag{22}$$

where $G(t)$ is a piecewise constant periodic function satisfying:

$$G(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ b & \text{if } 1 \le t < 2 \end{cases} \tag{23}$$

and $G(t) = G(t+2k)$, $k \in \mathbb{Z}$. In this case, the control is periodically switched on and off that corresponds to the case investigated in Insperger (2006).

The state at $t = T = 2$ can be expressed according to (17):

$$x(2) = \left(\mathrm{e}^{2a} + \int_1^2 \mathrm{e}^{a(2-s)} b\,\mathrm{e}^{a(s-1)}\mathrm{d}s\right)x(0) = \mathrm{e}^{2a}(1 + b\,\mathrm{e}^{-a})x(0). \tag{24}$$

Since (22) is a scalar equation, the coefficient of $x(0)$ is also the characteristic multiplier: $\mu = \mathrm{e}^{2a}(1 + b\,\mathrm{e}^{-a})$. For periodic systems, the decay ratio can be defined as $\rho = |\mu|^{1/T}$ that is equal to $\sqrt{|\mu|}$ in this case. The decay ratio can be used to compare act-and-wait control systems with different periods T to the autonomous system. The contour curves of ρ are presented in Fig. 9. The stability boundary corresponds to $\rho = 1$ that is denoted by thick lines.

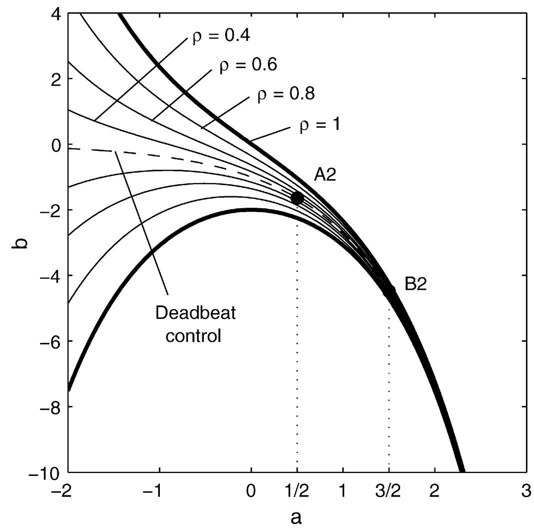

Fig. 9. Stability chart for (22) with (23)

It can be seen that if $b = -\mathrm{e}^{a}$ then $\mu = 0$ for any a. This means that deadbeat control can be attained for all system parameter a, even if the original autonomous system was unstable ($a > 1$). The optimal deadbeat control parameters for the cases $a = 1/2$ and $3/2$ are denoted by A2 and B2.

Fig. 10 shows the time histories for the autonomous system (18) at points A1, B1 and for the act-and-wait systems (22) with (23) at points A2 and B2. The simulation was performed using the semi-discretization method (Insperger and Stépán, 2002a). In the process of semi-discretization of time-periodic DDEs, the delayed terms are discretized while the undelayed terms are unchanged and the periodic coefficients are approximated by piecewise constant ones. The merit of the method is that it can effectively be used for constructing approximate Floquet transition matrix for periodic DDEs. In Fig. 10, thick lines denote the periods where the controller is on and thin lines

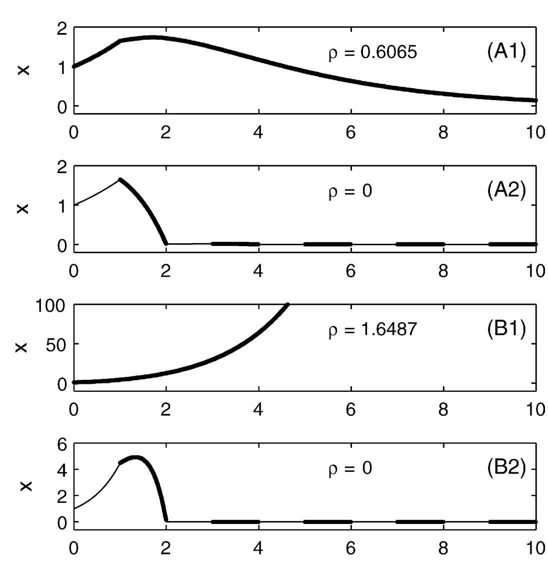

Fig. 10. Time histories for (18)(A1, B1) and for (22) with (23)(A2, B2).

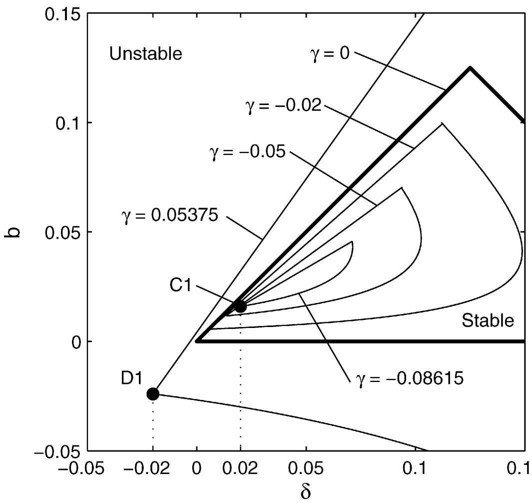

Fig. 11. Stability chart for (25).

denote the period where the controller is off. Panel A1 shows the stable autonomous system with the optimal control gain b resulting in a decay ratio $\rho = 0.6065$. Panel A2 shows the optimal (deadbeat) case for the act-and-wait control concept. It can clearly be seen that the system actually converges to zero within one period $T = 2$.

If $a = 3/2$, then the autonomous system is always unstable and it cannot be stabilized by any control gain b. The optimal case given by point B1 results in $\rho = 1.6487$, which corresponds to an unstable system. Using the act-and-wait control concept, the unstable plant can be stabilized, furthermore, in the optimal case, deadbeat control can be attained (see panel B2 in Fig. 10).

4.2. Second-order system

Consider the delayed oscillator equation:

$$\ddot{x}(t) + \delta x(t) = b x(t - 2\pi). \quad (25)$$

This equation corresponds to systems (7)–(9) with the matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\delta & 0 \end{bmatrix}, \qquad \mathbf{B} = 1, \qquad \mathbf{C} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\mathrm{T}}, \qquad \mathbf{D} = b. \quad (26)$$

The stability chart of (25)(see in Fig. (3)) is well known (see, e.g., Bhatt & Hsu, 1966; Stépán, 1989). Here, we will concentrate on the first triangle of this stability chart shown in Fig. 11.

Substitution of $\lambda = \gamma \pm i\omega$ into the characteristic equation

$$\lambda^2 + \delta - b\,\mathrm{e}^{-2\pi\lambda} = 0 \quad (27)$$

provides the γ-contour curves:

$$\delta = \omega^2 - \gamma^2 + \frac{2\gamma\omega\cos(2\pi\omega)}{\sin(2\pi\omega)}, \qquad b = \frac{-2\gamma\omega\,\mathrm{e}^{2\pi\gamma}}{\sin(2\pi\omega)}, \quad (28)$$

for $\omega > 0$ and the lines $b = (\gamma^2 + \delta)\mathrm{e}^{2\pi\gamma}$ for $\omega = 0$. The stability boundary corresponding to $\gamma = 0$ is denoted by thick lines in Fig. 11.

Two values of the parameter δ are considered:

$\delta = 0.02$: the optimal control gain is $b = 0.01596$, which results in a stable system with $\gamma = -0.08615$ and $\rho = \mathrm{e}^{\gamma} = 0.9175$. This case is denoted by point C1 in Fig. 11.

$\delta = -0.02$: the optimal control gain is $b = -0.02399$, which still results in an unstable system with $\gamma = 0.05375$ and $\rho = 1.0552$. This case is denoted by point D1 in Fig. 11.

Now, consider the act-and-wait control system:

$$\ddot{x}(t) + \delta x(t) = G(t) x(t - 2\pi). \quad (29)$$

where $G(t)$ is a piecewise constant periodic function satisfying:

$$G(t) = \begin{cases} 0 & \text{if } 0 \le t < 2\pi \\ b_1 & \text{if } 2\pi \le t < \frac{5}{2}\pi \\ b_2 & \text{if } \frac{5}{2}\pi \le t < 3\pi \end{cases} \quad (30)$$

and $G(t) = G(t + 3\pi k)$, $k \in \mathbb{Z}$. Here, $t_{\mathrm{w}} = \tau = 2\pi$, $t_{\mathrm{a}} = \pi$, $T = 3\pi$. The parameters b_1 and b_2 are the control parameters.

The 2×2 monodromy matrix of the system can be determined according to (17), and the stability chart can be constructed by calculating the characteristic multipliers. In Fig. 12, the contour curves of the decay ratio are shown as the function of the control parameters b_1 and b_2 for the cases $\delta = 0.02$ and -0.02. The stability boundaries corresponding to $\rho = 1$ are denoted by thick lines.

It can be seen that for both cases, the optimal control parameters denoted by points C2 and D2 result in deadbeat control with $\rho = 0$. If $\delta = 0.02$ then the optimal deadbeat parameters are $b_1 = -0.6129$ and $b_2 = 0.5141$ (point C2). If $\delta = -0.02$ then the system cannot be stabilized using an autonomous controller, but it can be stabilized using the act-and-wait concept, furthermore, the optimal parameters $b_1 = 0.9391$, $b_2 = -1.7345$ (point D2) result in deadbeat control. Although deadbeat control can clearly be attained, it should be mentioned that the corresponding parameter range is very narrow.

Fig. 13 shows the time histories for the autonomous system (25) at points C1, D1 and for the act-and-wait system (30) with (29) at points C2, D2. Thick lines denote the periods where the controller is on and thin lines denote the periods where the controller is off. Panel C1 shows the stable autonomous system with the optimal control gain b resulting in a decay ratio $\rho = 0.9175$. Panel C2 shows the optimal (deadbeat) case for the act-and-wait control concept. It can clearly be seen that the system actually converges to zero within the interval $2T = 6\pi$.

If $\delta = -0.02$, then the autonomous system is always unstable and it cannot be stabilized by any control gain b. The optimal case given by point D1 results in $\rho = 1.0552$, which corresponds to an unstable system (see panel D1 in Fig. 13). Panel D2 shows that the system can still be stabilized by using the act-and-wait control concept, and in the optimal case, deadbeat control can be attained.

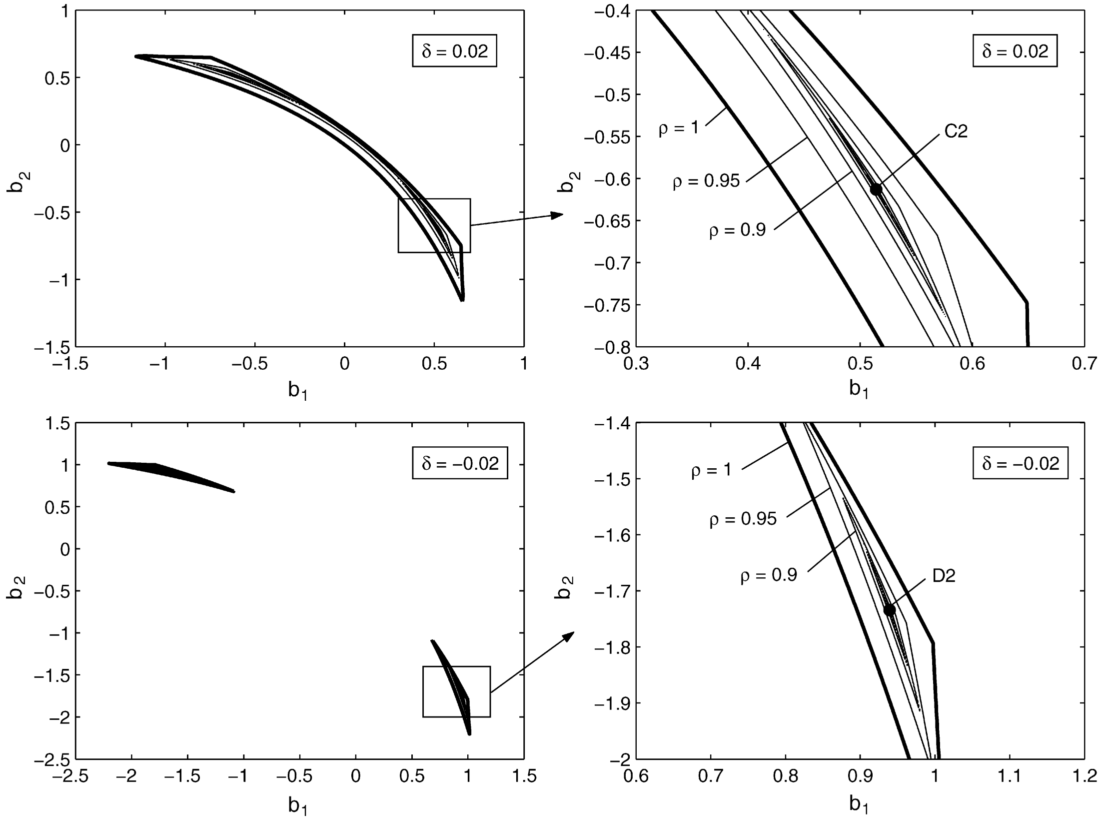

Fig. 12. Stability charts for (29) with (30).

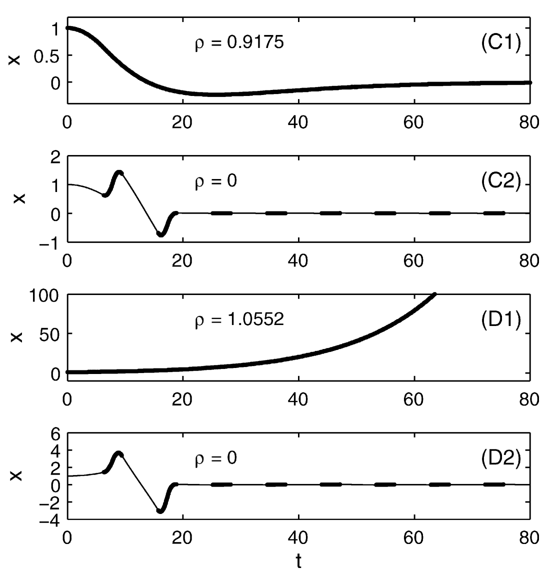

Fig. 13. Time histories for (25)(C1, D1) and for (29) with (30)(C2, D2).

5. Conclusion

Time delayed and time-periodic systems has not been considered often in engineering applications, since the corresponding analysis is rather complicated, and the available mathematical tools are not fully adopted in engineering. A unique analytical result for parametrically excited delayed systems is the stability chart of the delayed Mathieu equation. This chart serves as a reliable reference example to test numerical methods developed for time-periodic and also time-delayed systems. The study of these systems gives rise to new methods for stabilizing systems with feedback delay.

Time delay often arises in the feedback loop of control systems. The pole placement of such systems requires the control of infinitely many poles by a finite number of control parameters. To manage this problem, an effective way is the act-and-wait control method that is a special version of periodic controllers. The essence of the method is that the control gains are set to zero for an interval that is just larger than the time delay. Then the stabilization problem is reduced to a problem similar to the one posed by Brockett (1998) for ODEs. Two case studies are presented in order to demonstrate the efficiency of the method.

The main message is that the act-and-wait concept provides an alternative for control systems with feedback delays. The traditional approach is the continuous use of constant control gains according to the autonomous controller (9), when a cautious, slow feedback is applied with small gains resulting in slow convergence (if such controller can stabilize the system at all). The proposed alternative way is the act-and-wait control concept, when time-varying control gains are used in the acting phase and zero gains are used in the waiting intervals.

Several (actually, infinitely many) periodic functions could be chosen as time-periodic controllers. The main idea behind choosing the one that involves waiting intervals just longer than the feedback delay is that this kills the memory effect by waiting for the system's response induced by the previous action. Although it might seem unnatural not to actuate during the wait interval, act-and-wait concept is still a natural control logic for time-delayed systems. This is the way, for example, how one would adjust the shower temperature considering the delay between the controller (tap) and the sensed output (skin).

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