Simulation-based stability analysis method for state-dependent delay systems

Kristóf Martinovich * **and Dániel Bachrathy** *

*Department of Applied Mechanics, Faculty of Mechanical Engineering, Budapest University of Technology and Economics, HU, martinovich@mm.bme.hu

Summary. A well-established approach for analyzing the stability of delayed differential equations (DDE) is the semi-discretisation method, which approximates the monodromy operator and the characteristic multipliers through linear mapping. While powerful, it can be computationally expensive and hard to implement. An alternative method, the implicit subspace iteration (ISSI), builds upon the same principles but instead of a linear mapping based on a monodromy matrix, it relies only on numerical simulation. By integrating the initial state over the time period and performing pseudo-inverse calculations, the dominant multipliers can be iteratively computed. The advantage of this approach is its flexibility: any numerical solver can be used. Furthermore, it is applicable to difficult problems involving state-dependent delays (SDD), neutral, non-smooth or stiff systems, as long as the system can be numerically simulated. In our current work, we present the extended ISSI method to handle SDD systems. We compare the results of the ISSI with a linearised solution in a case study related to turning operations.

Mechanical model

A simplified model of the turning process consists of a rigid workpiece and a flexible tool modelled by a mass m, spring k and damper c. It already incorporates the regenerative effect through the force characteristics F_{x}, F_{y} which depend on the chip thickness $h(t)$. It is the difference between the tool position one revolution prior $y(t-\tau)$ and the current tool position $y(t)$, which results in a model of second-order DDE. The mathematical formula of the delay is defined by the following implicit integral equation

$$\int_{t-\tau}^{t} \Omega(t)\,\mathrm{d}t = 2\pi, \tag{1}$$

resulting in constant delay if the spindle speed is constant $\Omega(t) = \Omega_0$. We extended this model considering the motor characteristic by a PD controller which is intended to keep the nominally set Ω_0 spindle speed of the lathe. This configuration allows a small deviation $\omega(t)$ in rotational speed from the nominal value $\Omega(t) = \Omega_0 + \omega(t)$. Hence, the delay depends on one of the state variables $\tau(\omega_t)$ leading to a 2 DoF SDD model of the turning process illustrated in Figure 1. The linearisation of such systems is cumbersome, since the system is not differentiable with respect to the SDD, hence "true" linearisation is not possible. In [1], a comprehensive derivation was given to construct an associated linearised system calculating the Frechét derivate (the infinite-dimensional gradient of the delay). The linearised system can be given the following form

$$\dot{\mathbf{x}}(t) = \mathbf{L}\mathbf{x}(t) + \mathbf{R}\mathbf{x}(t-\tau), \tag{2}$$

where $\mathbf{L}$ is the Jacobian matrix, and $\mathbf{R}$ contains the coefficients of the derivatives by delayed variables and the additional terms from the Frechét derivative. Hereafter, the stability map may be calculated by substituting the trial solution resulting in the characteristic equation, then via D-separation, the nonlinear equation needs to be solved to obtain the stability boundaries.

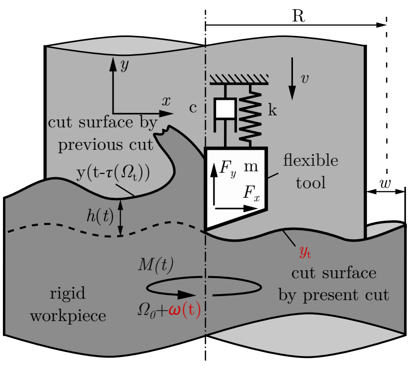

Figure 1: Turning model with state-dependent delay

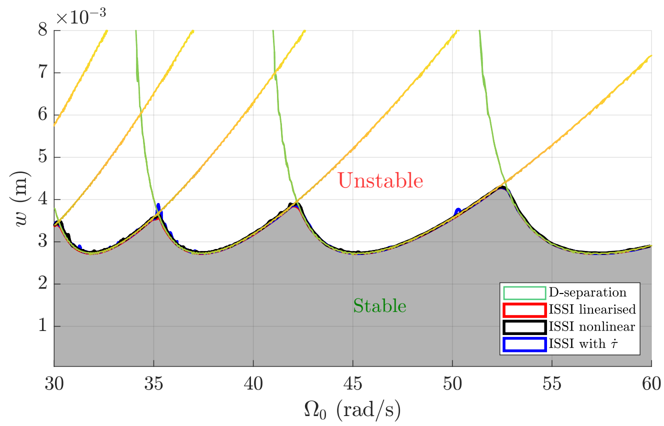

Figure 2: Comparision of the different linear stability boundary calculations

Numerical stability analysis methods

Semi-discretisaion

In the literature, a well-known technique for the stability analysis of DDEs is the semi-discretisation [2]. By its nature, a delayed system must be investigated in an infinite dimensional phase space, where the monodromy operator describes the transition of the history of the system to its state one period later. By discretising the history, the DDE can be approximated by a system of ordinary differential equations. It is constructed by a discrete linear mapping, where the Floquet transition matrix is calculated as a product of transition matrices related to each time step. The condition for the stability of the system is that all the eigenvalues of this matrix, the characteristic multipliers, must be less than one modulus. This technique can be computationally expensive due to many matrix multiplications (inversely proportional to the time step) for each parameter set, especially for high-resolution.

Implicit subspace iteration

The implicit subspace iteration method is based on the concept of semi-discretization [3]. But instead of the linear mapping which connects the initial state of the variables V to their state one period later S, it only relies on the numerical simulation of the equation of motion. Arbitrary many dominant eigenvalues and eigenvectors of the transition matrix can be approximated iteratively. Starting with a random initial state as a perturbation around the fixpoint, then via pseudo-inverse calculation the eigenvalues of the matrix are calculated as described in [4]. However, the fixpoint of the system must be known beforehand and considered in the calculation. Note, that the accuracy of the approximation depends on the number of iterations and the computed number of eigenvalues which are both case-sensitive and must be set cautiously. Moreover, the extent of the norm of the perturbation must be chosen carefully, as for nonlinear systems for too large perturbation the linear approximation might not be appropriate anymore (note, that SDD systems are nonlinear by their nature) and too small perturbation can lead to arithmetic errors.

Extended ISSI for SDD systems

In general, the iterated eigenvectors may be complex which can be handled by the numerical simulations but not in case of SDD systems. Considering (1), a complex variable would result in complex delay, but complex time is physically not interpretable. Hence, we extended the ISSI method by calculating the Schur decomposition of the approximated monodromy matrix. The Schur decomposition consists of a unitary matrix and a triangle matrix whose values are real. This transformation enables the use of the ISSI for SDD systems.

Case study

To validate the extended ISSI method, we compare the aforementioned fundamental, Frechét derivative based linearised system to three different implementations of the SDD turning model. The numerical simulation is implanted using the 4th-order Runge-Kutta method. First, the same linearised model is considered (2) as a baseline. Next, the original nonlinear system is used and at each time step, we search for the solution of (1) by approximating the integral and interpolating the delay. Lastly, the delay is rewritten into the following derivative form

$$\dot{\tau}(t) = \frac{\omega(t-\tau) - \omega(t)}{\Omega_0} \tag{3}$$

and an extended system of five DDEs is simulated which results in a significantly faster simulation. However, only the increment of the delay is calculated, hence generally, the accumulated error may cause inaccuracies. Despite this, the ISSI only requires a simulation for a short period of time where this effect is negligible.

Conclusions

The calculated stability boundaries are within line width to the Frechét derivative based associated linearised system as seen in Fig. 2. The extended ISSI is proven to be an efficient method to determine the linear stability properties of state-dependent delay systems. It is applicable even for very complex systems, such as SDD, neutral, non-smooth or stiff systems, as long as a proper time integration is possible.

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